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# The bifurcation set as a topological invariant for one-dimensional dynamics

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**Abstract**

For a continuous map on the unit interval or circle, we define the bifurcation set to be the collection of those interval holes whose surviving set is sensitive to arbitrarily small changes of (some of) their endpoints. By assuming a global perspective and focusing on the geometric and topological properties of this collection rather than the surviving sets of individual holes, we obtain a novel topological invariant for one-dimensional dynamics. We provide a detailed description of this invariant in the realm of transitive maps and observe that it carries fundamental dynamical information. In particular, for transitive non-minimal piecewise monotone maps, the bifurcation set encodes the topological entropy and strongly depends on the behavior of the critical points.

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(Some figures may appear in colour only in the online journal)

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## 1. Introduction

Given some dynamical system on a topological space and an open subset (called *hole* in the following), it is natural to study the associated *surviving set*, that is, the collection of all points which never enter this subset under forward iteration. In this framework, the theory of open dynamical systems is, for instance, concerned with escape rates, conditionally invariant measures and other closely related concepts, see for example [4, 10, 11, 17–19, 27, 32] for more information and further references. Recently, there has been an increased interest in understanding families of suitably parametrized interval holes of one-dimensional maps whose surviving sets fulfill certain properties, see for instance [1, 6, 12, 13, 20, 22, 23, 25, 26, 30, 35]. As a matter of fact, this thread of research goes back to the classical work by Urbański [36, 37].

In this spirit, we propose to study the family of all interval holes representing distinct surviving dynamics as a source of topological invariants. To be more precise, for a continuous map $f$ on the interval $[0, 1]$ or the circle $\mathbb{T}$, we consider the *bifurcation set* $\mathcal{B}_f$ which is given by all those intervals whose surviving set can change under arbitrarily small perturbations. To get a first impression of the bifurcation set, see figure 1 below, where an approximation of $\mathcal{B}_f$ for the doubling map on the circle is depicted.

Before we state our main results, let us introduce some basic definitions. Throughout this work, $\mathbb{I}$ refers to $[0, 1]$ (in which case we set $\partial\mathbb{I} = \{0, 1\}$) or $\mathbb{T}$ (in which case $\partial\mathbb{I} = \emptyset$). If $\mathbb{I} = [0, 1]$, a hole is given by an open interval $(a, b)$ with $a, b \in \mathbb{I}\backslash\partial\mathbb{I}$. [4] In this case, the collection of holes is naturally parametrized by

$$\Delta := \{(a, b) \in \mathbb{I} \times \mathbb{I} : a < b, \quad a, b \notin \partial\mathbb{I}\}.$$

If $\mathbb{I} = \mathbb{T}$, then a hole is an open interval of positive orientation from $a$ to $b$. In this case, the interval holes are naturally parametrized by the set

$$\Delta := \{(a, b) \in \mathbb{I} \times \mathbb{I} : a \neq b\}.$$

We denote the diagonal in $\mathbb{I} \times \mathbb{I}$ by $\Delta_0 := \{(a, a) : a \in \mathbb{I}\}$. Observe that $\Delta_0$ is explicitly *not* included in $\Delta$. If not stated otherwise, we consider $\Delta$ equipped with the subspace topology of the product topology on $\mathbb{I} \times \mathbb{I}$.

Now, consider a continuous map $f : \mathbb{I} \to \mathbb{I}$. The *surviving set* of $f$ with respect to $(a, b) \in \Delta$ is defined as

$$\mathcal{S}_f(a, b) := \{x \in \mathbb{I} : f^n(x) \notin (a, b) \quad \text{for all} \;\; n \geqslant 0\}.$$

Our main object of interest is the *bifurcation set* of $f$

$$\mathcal{B}_f := \{(a, b) \in \Delta : (x, y) \mapsto \mathcal{S}_f(x, y) \text{ is not locally constant in } (a, b)\}. \tag{1}$$

The geometric structure of $\mathcal{B}_f$ in $\Delta$ is constituted by a configuration of vertical and horizontal segments. Let us introduce some notation in order to describe it.

Given a closed subset $X \subseteq \Delta$, we define $\mathcal{H}(X)$ to be the family of non-trivial maximal horizontal line segments in $X$, and $\mathcal{V}(X)$ to be the family of non-trivial maximal vertical line

[4] The assumption that $a$ and $b$ avoid the boundary points $\{0, 1\}$ simply reduces certain technicalities and is not of any further importance. For an explicit study of general continuous maps on $[0, 1]$ with interval holes of the form $[0, t)$ and $(t, 1]$ where $t \in [0, 1]$, see [22].

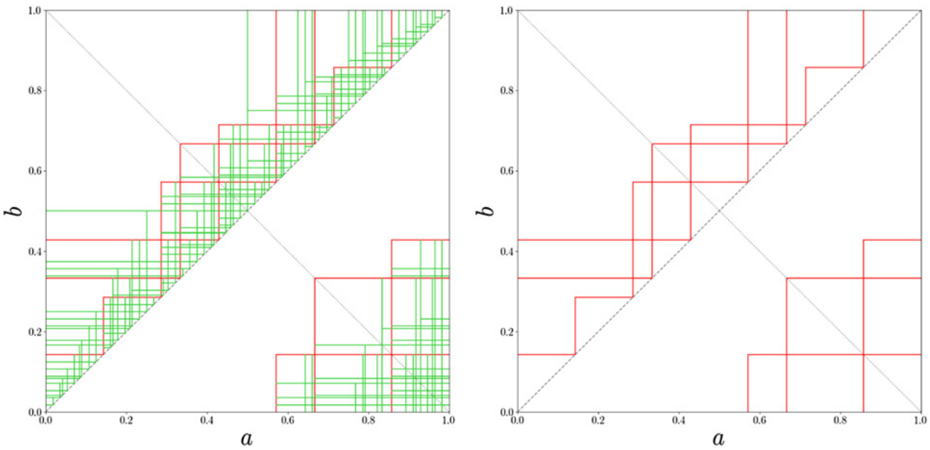



**Figure 1.** The left figure represents an approximation of the bifurcation set of the doubling map on the circle. The right figure shows the stairs of this bifurcation set corresponding to periodic orbits of period at most three. Moreover, let us point out that there is a natural relation between the description of all possible kneading sequences of expansive Lorentz maps and the bifurcation set of the doubling map, see [21, 24]. In particular, [24, figure 3] is also visible in the lower right corner of the left figure (after some minor adaptations).

segments in $X$. We define the set of *double points* $\mathcal{D}(X)$ to be the collection of points in $X$ which are in the intersection of an element of $\mathcal{H}(X)$ and an element of $\mathcal{V}(X)$. The set of *corner points* $\mathcal{C}(X) \subset \mathcal{D}(X)$ is given by those double points which are endpoints of an element of $\mathcal{H}(X)$ and of an element of $\mathcal{V}(X)$. Last, given $x \in \bigcup_{H\in\mathcal{H}(X)} H$ ($x \in \bigcup_{V\in\mathcal{V}(X)} V$) we denote the element of $\mathcal{H}(X)$ ($\mathcal{V}(X)$) containing $x$ by $H_x$ ($V_x$).

Double points will play an important part in retrieving dynamical information from the bifurcation set. In particular, this holds for corner points $x = (a_1, a_2) \in X$ whose coordinates are *links*, that is, there is an element in $\mathcal{H}(X)$ whose second coordinate coincides with $a_1$ and an element in $\mathcal{V}(X)$ whose first coordinate equals $a_2$. We refer to such an $x$ as a *step*. Given a step $x = (a_1, a_2) \in X$, we call the maximal collection of steps $F_x = \{\ldots, (a_1, a_2), (a_2, a_3), \ldots\} \subseteq \mathcal{C}(X)$, where for each element $y \in F_x$ there is a finite sequence $y = y_1, \ldots, y_n = x \in F_x$ such that $y_i$ shares a link with $y_{i+1}$ $(i = 1, \ldots, n-1)$, a *stair*. Note that $F_x$ is well defined and uniquely determined by $x$. Given $F_x = \{(a_1, a_2), \ldots, (a_{p-1}, a_p)\}$ is finite and $\mathbb{I} = [0, 1]$, we also refer to $a_1$ and $a_p$ as *terminal links*. The *length* of a stair is the cardinality of its links. Let us point out that the above terminology originates from the situation described in theorem A (b): for any step $x \in \mathcal{D}(\mathcal{B}_f)$ the segments $H_x$ and $V_x$ accumulate at the diagonal, so that the set $\bigcup_{y\in F_x}(H_y \cup V_y)$ resembles the shape of a stair (see also figure 1).

We can now state the first main assertion which is proven in section 3.

**Theorem A.** *Assume that $f : \mathbb{I} \to \mathbb{I}$ is continuous, transitive and not minimal. Then $\mathcal{B}_f$ is closed and the following hold.*

(*a*) *$\mathcal{B}_f \neq \emptyset$ and int$(\mathcal{B}_f) = \emptyset$.*

(*b*) *All elements of $\mathcal{H}(\mathcal{B}_f)$ and $\mathcal{V}(\mathcal{B}_f)$ accumulate at $\Delta_0$ and $\mathcal{B}_f = \bigcup_{H\in\mathcal{H}(\mathcal{B}_f)} H \cup \bigcup_{V\in\mathcal{V}(\mathcal{B}_f)} V$.*

(*c*) *$\mathcal{D}(\mathcal{B}_f)$ is closed and totally disconnected.*

(*d*) *Each endpoint of an element of $\mathcal{H}(\mathcal{B}_f)$ or $\mathcal{V}(\mathcal{B}_f)$ is in $\mathcal{D}(\mathcal{B}_f)$.*

(*e*) $\mathcal{B}_f$ *is path-connected.*

(*f*) *If* $(f_n)_{n\in\mathbb{N}}$ *is a sequence of continuous functions on* $\mathbb{I}$ *converging uniformly to* $f$, *then every accumulation point of* $\mathcal{B}_{f_n}$ (*w.r.t. the Hausdorff metric*) *is contained in* $\mathcal{B}_f$.

(*g*) *Every stair of length* $p$ *in* $\mathcal{B}_f$ *corresponds to a unique periodic orbit of period* $p$. *Furthermore, all but finitely many periodic orbits correspond to a stair.*

Observe that point (g) yields the important fact that periodic points and their periods can be identified in the bifurcation set. This in particular implies that the topological entropy for transitive non-minimal piecewise monotone maps can be deduced from the bifurcation set (see section 2.1 for more details).

Notice further that the second part of point (b) implies that the bifurcation set is a collection of horizontal and vertical segments, while the first part of (b) gives-together with point (d)-that these segments can essentially be obtained by drawing a horizontal and vertical line from the double points to the diagonal[5]. This observation emphasizes the importance of double points which is even more prominent due to their close relation to kneading sequences of expansive Lorentz maps (see the caption of figure 1) as well as of nice points introduced in [28] (see also remark 3.3).

As we will see, natural representatives of double points originate from the periodic points $\mathrm{Per}(f)$ and the preperiodic points of $f$ (see proposition 3.8 below). It turns out that periodic and preperiodic orbits are of general importance also beyond the associated double points. With theorem B, our second main result, we obtain assumptions which guarantee that already by drawing vertical and horizontal lines from points in the bifurcation set with one periodic or preperiodic coordinate and taking the closure of the respective union of segments in $\Delta$ *recovers* the bifurcation set (see remark 4.4 for more details).

In order to state theorem B, we need to introduce some further notation. Let $x = (a, b)$ be a corner point of $\mathcal{B}_f$. We say that $x$ is *isolated* in $\mathcal{B}_f$ whenever for some neighborhood $U$ of $x$ in $\Delta$ it holds

$$U \cap \mathcal{B}_f = U \cap (H_x \cup V_x).$$

Moreover, we call $x$ *isolated from below* whenever for some neighborhood $U$ of $x$ in $\Delta$ it holds for every $(a', b') \in \mathcal{B}_f \cap U\backslash(H_x \cup V_x)$ that

$$a' \in \mathbb{I}\backslash(a, b) \quad \text{or } b' \in \mathbb{I}\backslash(a, b).$$

Otherwise we call $x$ *accumulated from below*.

The next statement yields the sensitivity of $\mathcal{B}_f$ on the dynamical behavior of the critical points $\mathrm{Cri}(f)$ of $f$. We would like to remark that an essential ingredient of its proof are shadowing and stability properties of the surviving sets (see section 4).

**Theorem B.** *Suppose* $f : \mathbb{I} \to \mathbb{I}$ *is a continuous, transitive, not minimal and piecewise monotone map. Then the following hold.*

(*a*) *If* $\mathrm{Per}(f) \cap \mathrm{Cri}(f) = \emptyset$, *then every step is isolated from below. Moreover, in case* $\mathrm{Cri}(f)$ *is empty or contains only transitive points, we have that* $f$ *is a continuity point*[6] *of the bifurcation set and that* $\mathcal{B}_f$ *can be recovered from periodic and preperiodic points.*

[5] For $\mathbb{I} = \mathbb{T}$, this is true for all segments. For $\mathbb{I} = [0, 1]$, this is true for all but those lines in $\mathcal{B}_f$ with arbitrarily small first or second coordinate, see also the previous footnote.

[6] As in theorem A (f), we consider the space of all continuous maps $f : \mathbb{I} \to \mathbb{I}$ equipped with the uniform topology, and the space of all non-empty closed subsets of $\Delta$ endowed with the Hausdorff metric.

(*b*) *If* $\mathrm{Per}(f)\cap\mathrm{Cri}(f)\neq\emptyset$*, then there is at least one step accumulated from below or* $f$ *is a discontinuity point for the bifurcation set.*

Our last statement is an application of the above theorems and existing results concerning the family of restricted tent maps (see section 5 for the details). The presentation here is a simplified version of theorem 5.2.

**Theorem C.** *Let* $(T_s)_{s\in[\sqrt{2},2]}$ *be the family of restricted tent maps. Then there exist two disjoint and dense subsets of parameters, denoted by* $\mathcal{I}$ *and* $\mathcal{J}$ *where* $\mathcal{I}$ *has full measure, such that:*

(*a*) *For* $s\in\mathcal{I}$ *every step is isolated from below and* $s$ *is a continuity point of* $s\mapsto\mathcal{B}_{T_s}$.

(*b*) *For* $s\in\mathcal{J}$ *some step is accumulated from below and* $s$ *is a discontinuity point of* $s\mapsto\mathcal{B}_{T_s}$.

We close the introduction noting that although the bifurcation set itself is clearly not a dynamical invariant, we can easily introduce an induced invariant, see section 2.1. By means of this idea, each topological property of the bifurcation set turns into a topological invariant. This aspect as well as the relation with periodic orbits, topological entropy, and some measure theoretic aspects are further explained in section 2.

## 2. Interpretation of $\mathcal{B}_f$ and induced invariants

Consider a continuous map $f:\mathbb{I}\to\mathbb{I}$. Recall that the surviving set of $f$ with respect to the hole $(a,b)\in\Delta$ is given by

$$\mathcal{S}_f(a,b)=\{x\in\mathbb{I}: f^n(x)\notin(a,b)\quad\text{for all}\ \ n\geqslant 0\}=\bigcap_{n=0}^{\infty}f^{-n}(\mathbb{I}\backslash(a,b))$$

$$=\left(\bigcup_{n=0}^{\infty}f^{-n}(a,b)\right)^c.$$

Observe that surviving sets are forward invariant under $f$. We define the *bifurcation set* of $f$ by

$$\mathcal{B}_f:=\{(a,b)\in\Delta: a\in\mathcal{S}_f(a,b)\quad\text{or}\ \ b\in\mathcal{S}_f(a,b)\}. \tag{2}$$

Note that if $(a,b)\in\mathcal{B}_f$, both $a$ and $b$ may belong to $\mathcal{S}_f(a,b)$. Before proposition 2.2, we will comment on the difference between the above definition and (1).

We omit the obvious proof of the next statement (which was formulated for transitive maps in theorem A, already).

**Proposition 2.1.** *Let* $f$ *be a continuous self-map on* $\mathbb{I}$*. Then* $\mathcal{B}_f$ *is closed in* $\Delta$.

Clearly, if $(a,b)\in\mathcal{B}_f$ as defined in (2), then the surviving set of any hole containing $[a,b]$ does not contain $a$ and $b$ so that $(x,y)\mapsto\mathcal{S}_f(x,y)$ is not locally constant in $(a,b)$. Accordingly, $\mathcal{B}_f$ as defined in (2) is clearly contained in the collection given by (1). On the other hand, the collection from (1) is also contained in (and hence coincides with) that of (2), as the next proposition shows.

**Proposition 2.2.** *Suppose* $(a,b)$ *and* $(a',b')$ *are points in* $\Delta$ *belonging to the same connected component of* $\mathcal{B}_f^c$*. Then* $\mathcal{S}_f(a,b)=\mathcal{S}_f(a',b')$.

The proof of proposition 2.2 is a consequence of the next lemma. In what follows we set

$$\mathcal{S}_f^N(a,b) := \bigcap_{n=0}^{N} f^{-n}(\mathbb{I}\backslash(a,b))$$

and note that $\mathcal{S}_f(a,b) = \bigcap_{N\in\mathbb{N}}\mathcal{S}_f^N(a,b)$.

**Lemma 2.3.** *Suppose $(a,b) \in \mathcal{B}_f^c$. Then there is $\varepsilon > 0$ and $M \in \mathbb{N}$ such that*

$$\mathcal{S}_f^{N+2M}(a,b) \subseteq \mathcal{S}_f^{N+M}(a',b') \subseteq \mathcal{S}_f^N(a,b),$$

*for all $(a',b') \in B_\varepsilon(a,b)$ and all $N \in \mathbb{N}$.*

**Proof.** Note that for $(a,b)$ as in the assumptions, there is $\varepsilon > 0$ with $B_\varepsilon(a,b) \subseteq \mathcal{B}_f^c$ such that there are $\ell_a$ and $\ell_b$ in $\mathbb{N}$ with $f^{\ell_a}(B_\varepsilon(a)), f^{\ell_b}(B_\varepsilon(b)) \subseteq (a+\varepsilon, b-\varepsilon)$. Let $M := \max\{\ell_a, \ell_b\}$.

Consider $(a',b') \in B_\varepsilon(a,b)$ and suppose $a' \in (a,b)$ and $b' \notin (a,b)$ (the other cases work similarly). Trivially, $\mathcal{S}_f^{N+2M}(a,b) \subseteq \mathcal{S}_f^{N+2M}(a',b)$. Next, observe that if $x \in (a',b')\backslash(a',b)$, then by definition of $\ell_b$, we have $f^{\ell_b}(x) \in (a',b)$ and hence $\mathcal{S}_f^{N+2M}(a',b) \subseteq \mathcal{S}_f^{N+2M-\ell_b}(a',b') \subseteq \mathcal{S}_f^{N+M}(a',b')$. Likewise, we see that $\mathcal{S}_f^{N+M}(a',b') \subseteq \mathcal{S}_f^N(a,b')$ which clearly yields $\mathcal{S}_f^{N+M}(a',b') \subseteq \mathcal{S}_f^N(a,b)$. This finishes the proof. □

Observe that with $\varepsilon$ and $M$ as above, we hence have for all $(a_0,b_0)$ and $(a_1,b_1)$ in $B_\varepsilon(a,b) \subseteq \mathcal{B}_f^c$ and all $N \in \mathbb{N}$ that $\mathcal{S}_f^{N+4M}(a_0,b_0) \subseteq \mathcal{S}_f^{N+2M}(a_1,b_1) \subseteq \mathcal{S}_f^N(a_0,b_0)$. This immediately yields proposition 2.2.

It is immediate that $\mathcal{B}_f = \emptyset$ for $f : \mathbb{I} \to \mathbb{I}$ minimal. Notice that proposition 2.2 offers the converse of this statement[7]. This also yields the first part of point (a) of theorem A.

**Corollary 2.4.** *$\mathcal{B}_f = \emptyset$ if and only if $f$ is minimal.*

Recall that given a probability measure $\mu$ on $\mathbb{I}$, the (*exponential*) *escape rate* of a hole $(a,b)$ with respect to $\mu$ is defined as

$$\rho(\mu,(a,b)) := -\lim_{N\to\infty} \frac{1}{N} \log \mu(\mathcal{S}_f^N(a,b)).$$

If the above limit does not exist, we may likewise consider the *upper* and *lower escape rate* by considering the lim sup and liminf, respectively. For more information about escape rates and related concepts, see the references at the beginning of the introduction.

Another dynamical characterization of the bifurcation set is the following which is again a consequence of lemma 2.3.

**Corollary 2.5.** *For every probability measure $\mu$ on $\mathbb{I}$, the lower and upper escape rate are constant on each connected component of the complement of $\mathcal{B}_f$.*

Clearly, this result remains true when we consider non-exponential escape rates, too.

### *2.1. The bifurcation set as a strict invariant and deduced invariants*

In the following, we discuss different dynamical invariants involved with the bifurcation set and pose some naturally related questions.

[7] More precisely, in the interval case, proposition 2.2 yields transitivity for all points except 0 and 1. Yet, denseness of periodic points for transitive maps implies minimality of $f$, see the remark before proposition 3.12. Further, recall that there are no minimal continuous maps on $[0,1]$, i.e., $\mathcal{B}_f$ is always non-empty for $\mathbb{I} = [0,1]$.

First, let us assume that $f$ and $g$ are *conjugate*, i.e., $\pi \circ f = g \circ \pi$ where $\pi : \mathbb{I} \to \mathbb{I}$ is a homeomorphism. Then $\mathcal{B}_g = \{(\pi(a), \pi(b)) \in \Delta : (a, b) \in \mathcal{B}_f\}$ if $\pi$ is order preserving and $\mathcal{B}_g = \{(\pi(b), \pi(a)) \in \Delta : (a, b) \in \mathcal{B}_f\}$ otherwise. Hence, the bifurcation sets of conjugate maps are homeomorphic via a uniformly continuous self-homeomorphism on $\Delta$, where the uniform continuity is inherited from $\pi$. Now, for subsets $X, Y \subseteq \Delta$ we can define an equivalence relation by setting $X \sim Y$ if there is a uniformly continuous homeomorphism $p : \Delta \to \Delta$ with $p(X) = Y$. Then the equivalence class $[\mathcal{B}_f]$ defines a topological dynamical invariant for $f$.

**Question 1.** Are there natural families of maps where the bifurcation set is a complete topological invariant, that is, where homeomorphic bifurcation sets imply topological conjugacy?

Clearly, any topological property of $\mathcal{B}_f$ which is preserved under uniformly continuous homeomorphisms is a dynamical invariant of $f$. In a spirit similar to question 1, one may ask.

**Question 2.** Which dynamical invariants of transitive non-minimal one-dimensional dynamics can be obtained from the bifurcation set?

For example, if we start from $[\mathcal{B}_f]$, we can easily index the stairs and their lengths in $\mathcal{B}_f$. Accordingly, in the light of point (g) of theorem A, we can index the periodic orbits (all but finitely many if $\mathbb{I} = [0, 1]$) and their periods by an inspection of $[\mathcal{B}_f]$ for transitive maps. In particular, we can deduce for a transitive non-minimal piecewise monotone map $f : \mathbb{I} \to \mathbb{I}$ that its topological entropy $h(f)$ can be recovered from $\mathcal{B}_f$. For this recall that a continuous map $f : \mathbb{I} \to \mathbb{I}$ is called *piecewise monotone* if there are finitely many intervals $I_1, \ldots, I_n$ in $\mathbb{I}$ with $\mathbb{I} \subseteq \bigcup_{\ell=1}^{n} I_\ell$ such that $f$ is monotone on each $I_\ell$ (recall that $f$ is *monotone* on an interval $I \subset \mathbb{I}$ if $f|_I^{-1}(x)$ is connected for every $x \in I$). For this kind of maps we have that

$$h(f) \leqslant \limsup_{n\to\infty} \frac{1}{n} \log \#\{x \in \mathbb{I} : f^n(x) = x\},$$

see [29, corollaries 3 and 3']. Moreover, in remark 4.12 we explain that every transitive non-minimal piecewise monotone map is conjugate to a map with constant slope. This in turn implies that each monotone piece of $f$ intersects the diagonal at most once. Accordingly, we get $h(f) = \limsup_{n\to\infty} 1/n \log \#\{x \in \mathbb{I} : f^n(x) = x\}$ (see for example [2, p 218] for more details) and we obtain

$$h(f) = \limsup_{n\to\infty} \frac{1}{n} \log \#\{\text{stairs of length } n \text{ in } \mathcal{B}_f\}.$$

Another dynamical invariant visible in $\mathcal{B}_f$ for a continuous self-map $f$ on $\mathbb{I}$ is the group of automorphisms Aut$(f)$. These are all homeomorphisms $\pi : \mathbb{I} \to \mathbb{I}$ commuting with $f$, i.e., $f \circ \pi = \pi \circ f$. Each $\pi \in \mathrm{Aut}(f)$ defines a map $\hat{\pi} : \Delta \to \Delta$ mapping $(a, b)$ to $(\pi(a), \pi(b))$ or $(\pi(b), \pi(a))$ depending on whether $\pi$ is order preserving or reversing, respectively. Accordingly, we get that $\mathcal{B}_f$ is invariant under $\hat{\pi}$ and this means $\pi$ represents a certain symmetry of the bifurcation set. For an example of this observation, see figure 1, where the automorphism $\pi = -\mathrm{Id}$ of the doubling map is visible in the symmetry along the off-diagonal.

**Question 3.** Which symmetries of $\mathcal{B}_f$ originate from an automorphism of $f$?

While topological properties of $\mathcal{B}_f$ are preserved under conjugacy, we may still ask

**Question 4.** Is it possible to detect the existence of an infinite ergodic measure in $\mathcal{B}_f$?

The reader may recall that the Farey map is conjugate to the tent map where the former has an infinite absolutely continuous ergodic measure and the latter a finite one.

Discussing ergodic properties, let us briefly come back to the so-called nice points from [28] which were introduced to study possible ergodic behavior of $S$-unimodal maps on the interval. In particular, it is known that every $S$-unimodal map without periodic attractors has the weak-Markov property, which implies the non-existence of positive Lebesgue measure attracting Cantor sets. Nice points are essential for proving this assertion and a simple inspection of their definition shows that they can be derived from the bifurcation set.

We close this section with the following questions regarding possible generalizations of our main results:

**Question 5.** Does there exist a reasonable decomposition of $\mathcal{B}_f$ for continuous maps $f: \mathbb{I} \to \mathbb{I}$ which are not transitive?

**Question 6.** How do (finitely many) discontinuity points of $f$ affect the structure of $\mathcal{B}_f$?

**Question 7.** What is the effect of infinitely many critical points on the bifurcation set?

## 3. Proof of theorem A

In this section, we study the topology of the bifurcation set in general and for transitive systems in particular. We will obtain theorem A as a combination of several smaller propositions and lemmas proven in this part.

### 3.1. General properties of the bifurcation set

This section aims at a first understanding of basic topological properties of the bifurcation set.

For the sake of completeness, let us start by briefly recalling some standard notions from the theory of dynamical systems. For $f: \mathbb{I} \to \mathbb{I}$ and $x \in \mathbb{I}$ we refer to $\mathcal{O}(x) := \{f^n(x) : n \in \mathbb{N}_0\}$ as the *orbit* of $x$. If $\mathcal{O}(x)$ is finite, we call $x$ and likewise its orbit *preperiodic*. If $f^n(x) = x$ for some $n \in \mathbb{N}$, then $x$ as well as its orbit are referred to as *periodic* and we call $n$ a *period* of $x$. If $\overline{\mathcal{O}(x)} = \mathbb{I}$, that is, if $\mathcal{O}(x)$ is dense in $\mathbb{I}$, we say $x$ is *transitive*. We denote the collection of all periodic and transitive points of $f$ by $\mathrm{Per}(f)$ and $\mathrm{Tra}(f)$, respectively. If $\mathrm{Tra}(f) \neq \emptyset$, then we call $f$ *transitive* and if *Tra*$(f) = \mathbb{I}$, we say $f$ is *minimal*. It is well known and easy to see that $\mathrm{Tra}(f)$ is dense in $\mathbb{I}$ (residual, in fact) if $f$ is transitive. Finally, we call a subset $A \subseteq \mathbb{I}$ *$f$-invariant* if $A$ is closed and if $f(A) \subseteq A$. In case $A \subseteq \mathbb{I}$ is $f$-invariant and if there is an $x \in A$ with $\overline{\mathcal{O}(x)} = A$, we say $A$ is a *transitive* set.

Observe that the next statement yields point (f) of theorem A. In the following, we denote by $d$ the standard metric on $\mathbb{I}$ and by $d_\infty$ the *supremum metric* on the space of continuous self-maps on $\mathbb{I}$.

**Proposition 3.1.** *Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of continuous maps $f_n : \mathbb{I} \to \mathbb{I}$ which converges uniformly to $f: \mathbb{I} \to \mathbb{I}$. Then $\bigcap_{n\in\mathbb{N}} \overline{\bigcup_{k\geqslant n} \mathcal{B}_{f_k}} \subseteq \mathcal{B}_f$.*

**Proof.** Suppose $(a, b) \notin \mathcal{B}_f$. Then there is $\varepsilon > 0$ and $n, m \in \mathbb{N}$ with $f^n(a), f^m(b) \in (a + 3\varepsilon, b - 3\varepsilon)$. Choose $n_0$ sufficiently large so that $d_\infty(f_k^n, f^n), d_\infty(f_k^m, f^m) < \varepsilon$ for all $k \geqslant n_0$. By the triangle inequality and continuity of $f$, there is $\delta > 0$ such that $f_k^n(x) \in B_{2\varepsilon}(f^n(a))$ and $f_k^m(y) \in B_{2\varepsilon}(f^m(b))$ if $x \in B_\delta(a)$ and $y \in B_\delta(b)$. We may assume without loss of generality that $\delta < \varepsilon$. We have hence shown $(B_\delta(a) \times B_\delta(b)) \cap \bigcup_{k\geqslant n_0} \mathcal{B}_{f_k} = \emptyset$. Therefore, $\mathcal{B}_f^c \subseteq \left(\bigcap_{n\in\mathbb{N}} \overline{\bigcup_{k\geqslant n} \mathcal{B}_{f_k}}\right)^c$. □

We next turn to point (b) of theorem A. In the following, we say a set $V \subseteq \Delta$ *accumulates* at the diagonal $\Delta_0$ if $\inf_{(a,b)\in V} d(a,b) = 0$.

**Proposition 3.2.** *Let f be a continuous self-map on* $\mathbb{I}$*. For every point* $x \in \mathcal{B}_f$ *there exists an element* $H_x \in \mathcal{H}(\mathcal{B}_f)$ *with* $x \in H_x$ *or an element* $V_x \in \mathcal{V}(\mathcal{B}_f)$ *with* $x \in V_x$ *which accumulates at the diagonal* $\Delta_0$.

*Moreover, if f is transitive and* $(a, b)$ *is contained in an element of* $\mathcal{V}(\mathcal{B}_f)$ $(\mathcal{H}(\mathcal{B}_f))$*, then* $a \in \mathcal{S}_f(a, b)$ $(b \in \mathcal{S}_f(a, b))$*. In particular, each non-trivial maximal vertical or horizontal segment in* $\mathcal{B}_f$ *accumulates at* $\Delta_0$.

**Proof.** For the first part, suppose $x = (a, b) \in \mathcal{B}_f$ and assume without loss of generality that $a \in \mathcal{S}_f(a, b)$. Clearly, $a \in \mathcal{S}_f(a, b')$ for every $b' \in (a, b]$ which proves that there is a vertical segment in $\mathcal{B}_f$ which accumulates at $\Delta_0$ and contains $x$.

For the second part, we may assume without loss of generality to be given an element $V \in \mathcal{V}(\mathcal{B}_f)$. Denote by $\pi_2 : \Delta \to \mathbb{I}$ the canonical projection to the second coordinate. Given $(a, b) \in V$, let us assume for a contradiction that $a \notin \mathcal{S}_f(a, b)$. Then there is $n \in \mathbb{N}$ such that $f^n(a) \in (a, b)$. Now, there clearly is a transitive point $c \in \pi_2(V)$ with $c \in (f^n(a), b)$ or $b \in (f^n(a), c)$ and which-as its orbit is dense and thus hits $(a, c)$—is not in $\mathcal{S}_f(a, c)$. Therefore, $(a, c) \notin \mathcal{B}_f$ contradicting the assumption that $V \subseteq \mathcal{B}_f$. This proves the statement. $\square$

**Remark 3.3.** Observe that the previous statement implies that if $f$ is transitive, we have that $a, b \in \mathcal{S}_f(a, b)$ if and only if $(a, b) \in \mathcal{D}(\mathcal{B}_f)$.

**Corollary 3.4.** *Let f be a continuous transitive self-map on* $\mathbb{I}$*. Then* $\bigcup_{V\in\mathcal{V}(\mathcal{B}_f)} V$ *and* $\bigcup_{H\in\mathcal{H}(\mathcal{B}_f)} H$ *(and therefore* $\mathcal{D}(\mathcal{B}_f)$*) are closed.*

**Proof.** Let $(a_n, b_n)_{n\in\mathbb{N}}$ be a sequence of points in $\bigcup_{V\in\mathcal{V}(\mathcal{B}_f)} V$ (the case of $\bigcup_{H\in\mathcal{H}(\mathcal{B}_f)} H$ works similarly) converging to some $(a, b) \in \Delta$. By proposition 3.2, we know $(a_n, b_n)$ is contained in a vertical segment which accumulates at $\Delta_0$. Hence, for each $b' \in (a, b]$ we have a sequence $(a_n, b'_n)_{n\in\mathbb{N}}$ in $\bigcup_{V\in\mathcal{V}(\mathcal{B}_f)} V$ with $(a_n, b'_n) \to (a, b')$ as $n \to \infty$. Since $\mathcal{B}_f$ is closed (by proposition 2.1), we get $\{(a, b') : b' \in (a, b]\} \subseteq \mathcal{B}_f$, i.e., $(a, b) \in \bigcup_{V\in\mathcal{V}(\mathcal{B}_f)} V$ which finishes the proof. $\square$

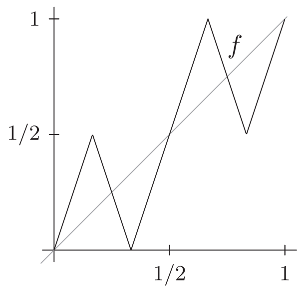
The second part of point (a) of theorem A is provided by

**Proposition 3.5.** *If* $f : \mathbb{I} \to \mathbb{I}$ *is transitive, then* $\mathrm{int}(\mathcal{B}_f) = \emptyset$.

**Proof.** Given $(a, b) \in \mathcal{B}_f$, we find arbitrarily close $(a', b')$ such that $a'$ and $b'$ are transitive points and hence $a', b' \notin \mathcal{S}_f(a', b')$. $\square$

Note that transitivity is not necessary in order to have $\mathrm{int}(\mathcal{B}_f) = \emptyset$. For example, on $\mathbb{I} = [0, 1]$, we may consider

$$f(x) := \begin{cases} 1/2 - 3\cdot|x - 1/6| & \text{if } x \in [0, 1/3], \\ 3\cdot(x - 1/3) & \text{if } x \in [1/3, 2/3], \\ 1/2 + 3\cdot|x - 5/6| & \text{if } x \in [2/3, 1]. \end{cases}$$



Here, $[0, 1/2]$ and $[1/2, 1]$ are transitive $f$-invariant subsets and we see, similarly as in the proof of proposition 3.5, that $\mathrm{int}(\mathcal{B}_f) = \emptyset$.

Recall that the set of *non-wandering points* of $f$ is defined by

$$\mathrm{NW}(f) := \{x \in \mathbb{I} : \forall\, \varepsilon > 0\ \exists n \in \mathbb{N} \quad \text{such that} \;\; f^n(B_\varepsilon(x)) \cap B_\varepsilon(x) \neq \emptyset\}.$$

We straightforwardly obtain

**Proposition 3.6.** *Let f be a continuous self-map on* $\mathbb{I}$*. If int*$(\mathcal{B}_f) = \emptyset$*, then* $\mathrm{NW}(f) = \mathbb{I}$.

Clearly, $\mathrm{NW}(f) = \mathbb{I}$ is not sufficient in order to have $\mathrm{int}(\mathcal{B}_f) = \emptyset$ as can be seen by considering the identity, for example.

### 3.2. Transitive case

The statements of the previous section suggest that the additional assumption of transitivity allows for a substantially more detailed description of the bifurcation set. With this observation in mind, we are now taking a closer look at the transitive case.

**Lemma 3.7.** *If* $f : \mathbb{I} \to \mathbb{I}$ *is continuous and transitive, then* $\mathcal{D}(\mathcal{B}_f)$ *is totally disconnected.*

**Proof.** Observe that since $\mathcal{D}(\mathcal{B}_f)$ is locally compact (see corollary 3.4), $\mathcal{D}(\mathcal{B}_f)$ is totally disconnected if and only if its topological dimensional is zero. For a contradiction, we assume that $\mathcal{D}(\mathcal{B}_f)$ is not zero dimensional so that there is $(a, b) \in \mathcal{D}(\mathcal{B}_f)$ such that $(a, b)$ does not have arbitrarily small clopen neighborhoods in $\mathcal{D}(\mathcal{B}_f)$. Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$ we have that the boundary of the rectangle $[a - \varepsilon, a + \varepsilon] \times [b - \varepsilon, b + \varepsilon]$ intersects $\mathcal{D}(\mathcal{B}_f)$ (note that we may assume without loss of generality that $\varepsilon_0 < 1/2 \cdot d(a, b)$). Observe that if $\mathcal{D}(\mathcal{B}_f)$ intersects one of the vertical sides of this boundary, this gives $v_\varepsilon \subseteq \mathcal{B}_f$ or $v^\varepsilon \subseteq \mathcal{B}_f$, where $v_\varepsilon$ and $v^\varepsilon$ are the vertical line segments $v_\varepsilon = \{a - \varepsilon\} \times (a - \varepsilon, b - \varepsilon_0]$ and $v^\varepsilon = \{a + \varepsilon\} \times (a + \varepsilon, b - \varepsilon_0]$, respectively. Likewise, if $\mathcal{D}(\mathcal{B}_f)$ intersects one of the horizontal sides, this implies $h_\varepsilon \subseteq \mathcal{B}_f$ or $h^\varepsilon \subseteq \mathcal{B}_f$, where $h_\varepsilon = [a + \varepsilon_0, b - \varepsilon) \times \{b - \varepsilon\}$ and $h^\varepsilon = [a + \varepsilon_0, b + \varepsilon) \times \{b + \varepsilon\}$. Hence,

$$[0, \varepsilon_0] = \bigcup_{\substack{\varepsilon\in[0,\varepsilon_0]\\ v_\varepsilon\subseteq\mathcal{B}_f}} \varepsilon \cup \bigcup_{\substack{\varepsilon\in[0,\varepsilon_0]\\ v^\varepsilon\subseteq\mathcal{B}_f}} \varepsilon \cup \bigcup_{\substack{\varepsilon\in[0,\varepsilon_0]\\ h_\varepsilon\subseteq\mathcal{B}_f}} \varepsilon \cup \bigcup_{\substack{\varepsilon\in[0,\varepsilon_0]\\ h^\varepsilon\subseteq\mathcal{B}_f}} \varepsilon.$$

According to corollary 3.4, the sets

$$\bigcup_{\substack{\varepsilon\in[0,\varepsilon_0]\\ v_\varepsilon\subseteq\mathcal{B}_f}} \varepsilon, \bigcup_{\substack{\varepsilon\in[0,\varepsilon_0]\\ v^\varepsilon\subseteq\mathcal{B}_f}} \varepsilon, \ldots$$

are closed. Hence by Baire's category theorem, we may assume without loss of generality that there is a non-degenerate interval $I \subseteq \bigcup_{\substack{\varepsilon\in[0,\varepsilon_0]\\ v_\varepsilon\subseteq\mathcal{B}_f}} \varepsilon$. But then $(a - I) \times (a, b - \varepsilon_0] \subseteq \mathcal{B}_f$ so that $\mathrm{int}(\mathcal{B}_f) \neq \emptyset$, contradicting proposition 3.5. □

Together with corollary 3.4, the previous statement proves point (c) of theorem A. We next consider point (d).

**Proposition 3.8.** *Suppose* $f : \mathbb{I} \to \mathbb{I}$ *is continuous and transitive. If* $(a, b)$ *is an endpoint of an element of* $\mathcal{H}(\mathcal{B}_f)$*, then* $(a, b) \in \mathcal{D}(\mathcal{B}_f)$ *and the orbit of b comes arbitrarily close to a. Likewise, if* $(a, b)$ *is an endpoint of an element of* $\mathcal{V}(\mathcal{B}_f)$*, then* $(a, b) \in \mathcal{D}(\mathcal{B}_f)$ *and the orbit of a comes arbitrarily close to b.*

**Proof.** We only consider $(a, b) \in \mathcal{H}(\mathcal{B}_f)$, the other case is similar. By the second part of proposition 3.2, we have to show that $a \in \mathcal{S}_f(a, b)$. Since $(a, b)$ is an endpoint of a maximal horizontal segment, the set $\{x \in \mathbb{I}\backslash[a, b] : (x, b) \in \mathcal{B}_f^c\}$ accumulates at $a$. By definition, for all $x$ in the previous set, there are positive integers $n_x$ and $m_x$, so that $f^{n_x}(b) \in (x, b)$ and $f^{m_x}(x) \in (x, b)$. As $b \in \mathcal{S}_f(a, b)$ (see proposition 3.2), this gives $f^{n_x}(b) \in (x, a]$ and hence $\inf_{n\in\mathbb{N}} d(f^n(b), a) = 0$. Thus, if there was $n \in \mathbb{N}$ with $f^n(a) \in (a, b)$ we would have that $f^m(b) \in (a, b)$ for some $m \in \mathbb{N}$ contradicting the fact that $b \in \mathcal{S}_f(a, b)$. Therefore, $a \in \mathcal{S}_f(a, b)$. □

Recall the definition of steps, links and stairs from the introduction. Given a transitive self-map on $\mathbb{I}$, it is easy to see that if $\{x_1, x_2, \ldots, x_p\}$ is a periodic orbit, then each pair of adjacent points $(x_{i_0}, x_{i_1})$ (where the interval $(x_{i_0}, x_{i_1})$ does not intersect the respective orbit) with $x_{i_0}, x_{i_1} \notin \partial\mathbb{I}$ is a step and all elements in $\{x_1, x_2, \ldots, x_p\}\backslash\partial\mathbb{I}$ are links. In this way, each periodic orbit with at least two elements not contained in $\partial\mathbb{I}$ is naturally associated to a stair in $\mathcal{B}_f$. In fact, we have the following

**Proposition 3.9.** *Given $f : \mathbb{I} \to \mathbb{I}$ is continuous and transitive, every stair of $\mathcal{B}_f$ is of finite length and realized by a unique periodic orbit.*

**Proof.** Assume we are given a stair of length $p \in \mathbb{N} \cup \{\infty\}$. By definition, each element $(x_i, x_{i+1})$ of the stair is a corner point so that $x_i, x_{i+1} \in \mathcal{S}_f(x_i, x_{i+1})$, due to proposition 3.8. Further, as $x_i$ is a link, there is an element of $\mathcal{H}(\mathcal{B}_f)$ which accumulates at $(x_i, x_i) \in \Delta_0$, so that proposition 3.2 yields that $x_i \in \mathcal{S}_f(c, x_i)$ for some $c < x_i$. Hence, the orbit of $x_i$ does not hit the set $(c, x_i) \cup (x_i, x_{i+1})$ and can therefore not accumulate at $x_i$. Likewise, we obtain that the orbit of $x_{i+1}$ cannot accumulate at $x_{i+1}$. However, due to proposition 3.8, the orbit of $x_i$ comes arbitrarily close to $x_{i+1}$ and the orbit of $x_{i+1}$ comes arbitrarily close to $x_i$. This clearly yields that $x_i$ is an iterate of $x_{i+1}$ and vice versa. Hence, $x_i$ and $x_{i+1}$ are elements of a periodic orbit. We conclude that all links associated to a stair come from one and the same periodic orbit of period not bigger than $p + 2$. This proves thestatement. □

**Corollary 3.10.** *Let $f : \mathbb{I} \to \mathbb{I}$ be continuous and transitive. Then, for all but finitely many $p \geqslant 2$, there is a one-to-one correspondence between periodic orbits of minimal period p and stairs of length p.*

**Proof.** By the above, there is a one-to-one correspondence between stairs and periodic orbits which contain at least two elements within $\mathbb{I}\backslash\partial\mathbb{I}$. Further, unless a given periodic orbit hits $\partial\mathbb{I}$, its period obviously coincides with the length of the associated stair. As there are at most two periodic orbits which hit $\partial\mathbb{I}$, the statement follows. □

**Remark 3.11.** We would like to stress that in case of $\mathbb{I} = \mathbb{T}$, it is straightforward to see that the above one-to-one correspondence holds true for all periods $p \geqslant 2$, in fact.

Slightly abusing notation, given a step $x$, we may also refer to the point-set $S_x = V_x \cup H_x \subseteq \mathcal{B}_f$ as a *step*. In a similar fashion, given a stair $F_x$, we may also refer to the union of all maximal vertical and horizontal segments whose first and second coordinate, respectively, coincides with a link of $F_x$ as the stair $F_x$. Notice that for $\mathbb{I} = [0, 1]$, this union not only includes all respective steps (considered as point-sets) but also the horizontal and vertical segments associated to terminal links. We may refer to these segments as *terminal segments* of $F_x$. Observe that since each stair is realized by a periodic orbit, the terminal segments accumulate at $\{0\} \times \mathbb{I}$ and $\mathbb{I} \times \{1\}$.

By a *path* in $\mathcal{B}_f$, we refer to a continuous map $\gamma : [0, 1] \to \Delta$ with $\gamma([0, 1]) \subseteq \mathcal{B}_f$. Recall that $\mathcal{B}_f$ is *path-connected* if for all $x, y \in \mathcal{B}_f$ there is a path $\gamma$ in $\mathcal{B}_f$ from $x$ to $y$, that is, $\gamma(0) = x$

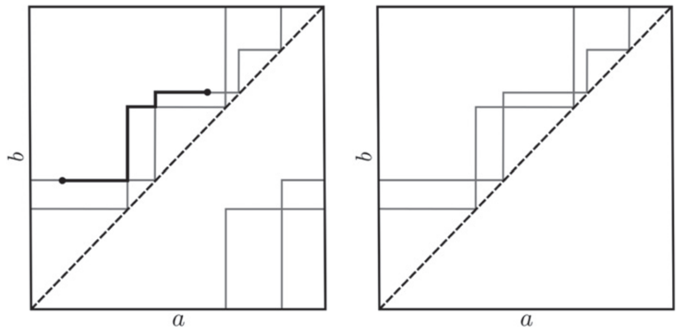



**Figure 2.** Stairs illustrated for maps on the circle (left) and on the interval (right). The left figure also depicts a path (bold line) as discussed in proposition 3.12.

and $\gamma(1) = y$. In order to prove the path-connectedness of $\mathcal{B}_f$, we make use of the following observation whose proof is based on the classical fact that a continuous transitive and non-minimal self-map on $\mathbb{I}$ has a dense set of periodic points (for interval maps, see [33] and also [7, lemma 41 on p 156]; for maps on the circle, this follows from [15, theorem A] together with [5, corollary 2]).

**Proposition 3.12.** *Suppose $f : \mathbb{I} \to \mathbb{I}$ is continuous and transitive. Given two points $(a, b)$ and $(a', b')$ on a stair $F_x$ (considered as the above union of segments), there is a continuous path in $\mathcal{B}_f$ from $(a, b)$ to $(a', b')$.*

**Proof.** We may assume without loss of generality that $(a, b)$ and $(a', b')$ lie on neighboring steps, that is, $(a, b) \in S_{(y_1,y_2)}$ and $(a', b') \in S_{(y_2,y_3)}$ for some $(y_1, y_2), (y_2, y_3) \in F_x$ (note that if $(a, b)$ or $(a', b')$ lies on a terminal segment, the following proof works exactly the same). As $f$ is transitive, there is a transitive point $y \in (y_1, y_2)$. By transitivity of $y$, there is $n \in \mathbb{N}$ such that $f^n(y) \in (y_2, y_3)$. Clearly, for a small enough interval $J \subseteq (y_1, y_2)$ containing $y$, we have $f^n(J) \subseteq (y_2, y_3)$. By denseness of periodic points, there is a periodic point $z \in J$. Let $z_1$ and $z_2$ be those points in the orbit of $z$ which are the furthest to the right in $\mathcal{O}(z) \cap (y_1, y_2)$ and the furthest to the left in $\mathcal{O}(z) \cap (y_2, y_3)$, respectively. Clearly, $(z_1, z_2)$ is a step and $S_{(z_1,z_2)}$ intersects both $S_{(y_1,y_2)}$ and $S_{(y_2,y_3)}$. Let $\gamma_1$ be some path in $S_{(y_1,y_2)}$ from $(a, b)$ to the unique intersection point $(c, d)$ of $S_{(y_1,y_2)}$ and $S_{(z_1,z_2)}$; let $\gamma_2$ be a path in $S_{(z_1,z_2)}$ from $(c, d)$ to the unique intersection point $(c', d')$ of $S_{(z_1,z_2)}$ and $S_{(y_2,y_3)}$; let $\gamma_3$ be a path in $S_{(y_2,y_3)}$ from $(c', d')$ to $(a', b')$. Clearly, the concatenation of $\gamma_1$, $\gamma_2$ and $\gamma_3$ is a path in $\mathcal{B}_f$ from $(a, b)$ to $(a', b')$. □

We next obtain point (e) of theorem A.

**Lemma 3.13.** *If $f : \mathbb{I} \to \mathbb{I}$ is continuous and transitive, then $\mathcal{B}_f$ is path-connected.*

**Proof.** We first observe that given two points $x$ and $y$ on stairs $F_x$ and $F_y$, respectively, there is a path in $\mathcal{B}_f$ from $x$ to $y$. To see this, it suffices—due to the previous statement—to show that there is a non-empty intersection between some segment associated to $F_x$ and some segment associated to $F_y$. This, however, follows immediately from the fact that on $\mathbb{I} = \mathbb{T}^1$, each stair wraps around $\Delta_0$ while on $\mathbb{I} = [0, 1]$, the horizontal and vertical terminal segment of each stair accumulates at $\{0\} \times \mathbb{I}$ and $\mathbb{I} \times \{1\}$, respectively (see figure 2).

Now, suppose we are given arbitrary points $x, y \in \mathcal{B}_f$. Due to proposition 3.2, we may assume without loss of generality that $x = (a, b)$ lies on a non-trivial horizontal segment $H$. Due to the denseness of periodic points, we find a periodic point $c \in \mathbb{I}$ with $c \in (a, b)$. Without loss of generality, we may assume that $\mathcal{O}(c)$ contains at least two points in $\mathbb{I}\backslash\partial\mathbb{I}$. Choose $c'$ to be the right-most point in $\mathcal{O}(c) \cap (a, b)$. Then, the vertical segment (terminal or not) of the stair associated to $\mathcal{O}(c)$ which accumulates at $(c', c')$ clearly intersects $H$. Hence, there is a path $\gamma_1$ from $x$ to a point $z_x$ on a stair $F_{z_x}$ in $\mathcal{B}_f$. Likewise, we obtain a path $\gamma_2$ from $y$ to a point $z_y$ on a stair $F_{z_y}$ in $\mathcal{B}_f$ whose inverse (from $z_y$ to $y$) we denote by $\overline{\gamma_2}$. By the above observation, there is a path $\gamma_3$ in $\mathcal{B}_f$ from $z_x$ to $z_y$. Altogether, the concatenation $\overline{\gamma_2} \cdot \gamma_3 \cdot \gamma_1$ is a path in $\mathcal{B}_f$ from $x$ to $y$ which proves the statement. $\square$

## 4. Proof of theorem B

In this section, we turn to the problem of identifying critical points and their dynamical behavior by means of the bifurcation set. For this recall that given a continuous map $f : \mathbb{I} \to \mathbb{I}$, $x \in \mathbb{I}$ is referred to as a *critical point* (alternatively *turning point*) if there is no neighborhood of $x$ on which $f$ is monotone. The collection of all critical points of $f$ is denoted by $\mathrm{Cri}(f)$. Observe that $\mathrm{Cri}(f)$ is obviously closed.

Let us point out that theorem B follows from theorem 4.11 (the main result of this section), see remark 4.12.

### 4.1. Implications of hyperbolicity

Besides transitivity, we will impose additional assumptions on the map $f$. In particular, we will assume certain forms of hyperbolicity. As we are dealing with results of a topological flavor, we consider the following definition of hyperbolicity: an $f$-invariant set $A \subseteq \mathbb{I}$ is referred to as *hyperbolic* for a continuous map $f : \mathbb{I} \to \mathbb{I}$ and a compatible metric $d$ if there exist $\varepsilon > 0$ and $\lambda > 1$ and an open neighborhood $U$ of $A$ such that $d(f(x), f(y)) > \lambda \cdot d(x, y)$ for all $x, y \in U$ with $d(x, y) < \varepsilon$. In this case, we may also say that $f$ is *$\varepsilon$-locally $\lambda$-expanding on $U$* (with respect to $d$). Note that a smooth map $f : \mathbb{I} \to \mathbb{I}$ which is hyperbolic on an invariant set $A$ in the classical sense is also hyperbolic in the above sense with respect to some metric $d$ equivalent to the usual one (see for instance the proof of theorem 2.3 in chapter 3 of [16]). Henceforth, all metrics are considered to be equivalent to the standard metric on $\mathbb{I}$ and throughout denoted by $d$.

We call $x \in \mathbb{I}$ *hyperbolic* if $\overline{\mathcal{O}(x)}$ is hyperbolic in the above sense. Notions like *hyperbolic steps* or *hyperbolic double points* are defined in the natural way.

Suppose $x \in \mathbb{I}$ is a periodic point of $f : \mathbb{I} \to \mathbb{I}$ with minimal period $p$. We say that $f$ *preserves orientation at $a \in \mathcal{O}(x)$* whenever $f^p|_J$ preserves orientation in some neighborhood $J$ of $a$. Otherwise, we say that $f$ *reverses orientation at $a$*. Given $a, b \in \mathcal{O}(x)$, we denote by $n_{a,b}$ the minimum time for going from $a$ to $b$ by iteration of $f$. We say that $f$ *preserves orientation from $a$ to $b$* whenever $f^{n_{a,b}}|_J$ preserves orientation in some neighborhood $J$ of $a$. Otherwise, we say that $f$ *reverses orientation from $a$ to $b$*.

Concerning the next statement, recall that due to proposition 3.9 every step is associated to a periodic point. We may hence refer to the period of this periodic point also as the *period* of the respective step.

**Lemma 4.1.** *Let $f : \mathbb{I} \to \mathbb{I}$ be continuous and transitive. Suppose $(a, b) \in \mathcal{B}_f$ is a hyperbolic step of period $p$. The following holds.*

(*a*) *If $f$ reverses orientation at $a$ or $b$, then $(a, b)$ is an isolated corner point of $\mathcal{B}_f$.*

(*b*) *If $f$ preserves orientation both at $a$ and $b$, then $(a, b)$ is isolated from below.*

**Proof.** We start by proving (a). For a contradiction, suppose there is $(a',b') \in \mathcal{B}_f \setminus (V_{(a,b)} \cup H_{(a,b)})$ arbitrarily close to $(a,b)$. Without loss of generality, we may assume $a' \in \mathcal{S}_f(a',b')$.

First, consider $a' = a$. Then we necessarily have $b' \in [a,b]^c$ (since otherwise we had $(a',b') \in V_{(a,b)}$) and thus $(a',b') \supsetneq (a',b)$. As the orbit of $a' = a$ accumulates at $b$ (see proposition 3.8), this gives $a' \notin \mathcal{S}_f(a',b')$.

Now, consider $a' \neq a$ and assume without loss of generality that $a' \in (a,b)$ (the other case can be dealt with similarly). Assuming that $(a',b')$ is sufficiently close to $(a,b)$, we have $f^{2p}(a') \in (a',b')$ since $f$ is expanding in a neighborhood of $\mathcal{O}(a)$ (as it is hyperbolic on $\mathcal{O}(a)$) and $f^{2p}$ is order-preserving in a neighborhood of $a$. Hence, $a' \notin \mathcal{S}_f(a',b')$. This contradicts the assumptions on $a'$ and finishes the proof of the first part.

Let us now turn to part (b). Assume for a contradiction that there is $(a',b') \in \mathcal{B}_f$ arbitrarily close to $(a,b)$ with $[a',b'] \subseteq (a,b)$. As $f^p$ is expanding and order preserving both in $a$ and $b$, we have $f^p(a'), f^p(b') \in (a',b')$ if $a'$ and $b'$ are sufficiently close to $a$ and $b$. Hence, $a',b' \notin \mathcal{S}_f(a',b')$ which contradicts the assumptions. $\square$

**Remark 4.2.** Assume the situation of the previous statement. It is not hard to see that if $f$ preserves orientation at $a$ and $b$ and additionally preserves orientation from $a$ to $b$, then $(a,b)$ is actually isolated. In particular, if $f$ is uniformly expanding, every step is isolated.

Recall that given $x \in \mathbb{I}$, its *$\omega$-limit set* $\omega_f(x)$ is defined to be the collection of all accumulation points of $\mathcal{O}(x)$. It is well known and easy to see that $\omega_f(x)$ is non-empty, compact and contains *recurrent points*, that is, there is $y \in \omega_f(x)$ such that $y \in \omega_f(y)$.

We call a double point $(a,b) \in \mathcal{B}_f$ *(pre)periodic*, if both $a$ and $b$ are (pre)periodic. The proof of the next statement makes use of standard shadowing arguments.

**Lemma 4.3.** *Let $f: \mathbb{I} \to \mathbb{I}$ be continuous and transitive. Consider $(a,b) \in \mathcal{B}_f$ with $a \in \mathcal{S}_f(a,b)$ and $b \notin \mathcal{S}_f(a,b)$ and suppose the orbit of $a$ is hyperbolic. If $a$ is not preperiodic, then $(a,b)$ is accumulated by points of the form $(\tilde{a},b) \in \mathcal{B}_f$ with $\tilde{a}$ preperiodic, $\mathcal{O}(\tilde{a})$ hyperbolic and $\tilde{a} \in \mathcal{S}_f(\tilde{a},b)$. A similar statement holds if we interchange the roles of $a$ and $b$.*

*Moreover, if $(a,b) \in \mathcal{B}_f$ is a double point which is not preperiodic and the orbits of $a$ and $b$ are hyperbolic, then $(a,b)$ is accumulated by hyperbolic preperiodic double points.*

**Proof.** Let $a \in \mathcal{S}_f(a,b)$ (the other case is similar) and assume $a$ is not preperiodic. Due to the assumptions, there is an open set $U$ (and a compatible metric $d$) with $\mathcal{O}(a) \cup \omega_f(a) \subseteq U$ such that $f$ is $\delta$-locally $\lambda$-expanding on $U$. Without loss of generality, we may assume that $\delta > 0$ is such that $B_\delta(x) \subseteq U$ for all $x \in \mathcal{O}(a) \cup \omega_f(a)$.

Choose some $\varepsilon < \delta/2$ and let $c \in \omega_f(a)$ be a recurrent point. Pick $n \in \mathbb{N}$ with $d(f^n(c),c) < \varepsilon$. Since $f$ is $\delta$-locally $\lambda$-expanding on $U$, we may assume without loss of generality that $n$ is large enough to ensure that $f^n(B_\varepsilon(c)) \supseteq B_{2\varepsilon}(f^n(c))$. Choose $I$ to be the connected component of $\bigcap_{\ell=1}^n f^{-\ell}\left(B_{2\varepsilon}(f^\ell(c))\right) \cap B_\varepsilon(c)$ which contains $c$.

By the assumptions on $n$, we have $f^n(I) = B_{2\varepsilon}(f^n(c)) \supseteq B_\varepsilon(c) \supseteq I$. Hence, there is a periodic point $d \in I$ of period $n$ whose orbit is $2\varepsilon$-close to $\omega_f(a)$ (by definition of $I$). Since $f$ is $\delta$-locally $\lambda$-expanding on $U$, there further is $m \in \mathbb{N}$ and a point $a' \in \mathbb{I}$ such that $f^m(a') = d$ and $\max_{\ell=0,\ldots,m-1} d(f^\ell(a'), f^\ell(a)) < 2\varepsilon$. Set $\tilde{a}$ to be the right-most point of $\mathcal{O}(a') \cap B_{2\varepsilon}(a)$.

Let $b \notin \mathcal{S}_f(a,b)$. Then $\mathcal{O}(a)$ is at positive distance to $b$ (otherwise $a$ would not survive) and we may assume $\varepsilon > 0$ to be small enough to ensure that $a$ does not come $2\varepsilon$-close to $b$ so that $\mathcal{O}(\tilde{a}) \cap (\tilde{a},b) = \emptyset$, i.e., $\tilde{a} \in \mathcal{S}_f(\tilde{a},b)$. As $\varepsilon$ can be chosen arbitrarily small, the first part follows.

Next, let us assume $b \in \mathcal{S}_f(a,b)$, that is, $(a,b)$ is a double point. If $\mathcal{O}(\tilde{a}) \cap B_{2\varepsilon}(b) \neq \emptyset$, set $\tilde{b}$ to be the left-most point in $\mathcal{O}(\tilde{a}) \cap B_{2\varepsilon}(b)$. Then, $(\tilde{a},\tilde{b})$ is preperiodic and moreover a double point with $d(\tilde{a},a), d(\tilde{b},b) < 2\varepsilon$. If $\mathcal{O}(\tilde{a}) \cap B_{2\varepsilon}(b) = \emptyset$, then $(\tilde{a},b)$ is a double point. If $b$ is preperiodic, $(\tilde{a},b)$ is hence a preperiodic double point $2\varepsilon$-close to $(a,b)$. If $b$ is not preperiodic and

$\mathcal{O}(\tilde{a}) \cap B_{2\varepsilon}(b) = \emptyset$, repeat the above argument for $(\tilde{a}, b)$ with the roles of $a$ and $b$ interchanged. In all cases, we end up with a preperiodic double point $(\tilde{a}, \tilde{b})$ with $d(\tilde{a}, a), d(\tilde{b}, b) < 4\varepsilon$. Since $\varepsilon > 0$ can be chosen arbitrarily small, this finishes the proof. □

**Remark 4.4.** Lemma 4.3 allows us to formulate conditions under which the bifurcation set can be obtained from (genuinely) smaller subsets. We say $\mathcal{R} \subseteq \mathcal{B}_f$ *recovers* the bifurcation set if

$$\mathcal{B}_f = \overline{\bigcup_{x \in \mathcal{R}} H_x \cup V_x}.$$

Due to lemma 4.3, in case every transitive subset without critical points is hyperbolic, the set of points $(a, b) \in \mathcal{B}_f$ with $a$ or $b$ preperiodic recovers $\mathcal{B}_f$. In particular, this can be ensured for a continuous and transitive piecewise uniformly expanding map $f : \mathbb{I} \to \mathbb{I}$ where $\mathrm{Cri}(f)$ is empty or consists of transitive points only.

Given $(a, b) \in \mathcal{B}_f$, we call $\{a, b\} \cap \mathcal{S}_f(a, b)$ the *surviving endpoints* of $(a, b)$.

**Theorem 4.5.** *Let $f : \mathbb{I} \to \mathbb{I}$ be continuous and transitive and suppose every critical point of $f$ is transitive. Assume further that every transitive invariant subset of $\mathbb{I}$ which does not contain a critical point is hyperbolic. Then the mapping $g \mapsto \mathcal{B}_g \in 2^{\Delta}$ is continuous at $f$ with respect to the uniform topology on the space of continuous self-maps on $\mathbb{I}$ and the Hausdorff metric on $2^{\Delta}$.*

**Proof.** According to proposition 3.1, given a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous maps $f_n : \mathbb{I} \to \mathbb{I}$ with $f_n \to f$ uniformly as $n \to \infty$, it suffices to show that for each $\varepsilon > 0$ there is $n_0$ such that for all $n \geqslant n_0$ we have $B_\varepsilon\left(\mathcal{B}_{f_n}\right) \supseteq \mathcal{B}_f$.

Pick $\varepsilon > 0$. Observe that due to proposition 2.1, $\mathcal{B}_f$ is precompact so that there are finitely many $(a_1, b_1), \ldots, (a_M, b_M) \in \mathcal{B}_f$ with $\mathcal{B}_f \subseteq \bigcup_{j=1}^M B_\varepsilon\left((a_j, b_j)\right)$. As the elements of $\mathrm{Cri}(f)$ are transitive, we further have that the surviving endpoints of $(a_1, b_1), \ldots, (a_M, b_M)$ are not critical and hence, at a positive distance to $\mathrm{Cri}(f)$ due to the compactness of $\mathrm{Cri}(f)$.

By the assumptions, this yields that the orbits of the surviving endpoints are hyperbolic. Hence, by lemma 4.3, there are $(\tilde{a}_1, \tilde{b}_1), \ldots, (\tilde{a}_M, \tilde{b}_M) \in \mathcal{B}_f$ with $\mathcal{B}_f \subseteq \bigcup_{j=1}^M B_{2\varepsilon}\left((\tilde{a}_j, \tilde{b}_j)\right)$ and such that at least one of the surviving endpoints of each pair among $(\tilde{a}_1, \tilde{b}_1), \ldots, (\tilde{a}_M, \tilde{b}_M)$ is preperiodic and hyperbolic. We denote these surviving endpoints by $y_1, \ldots, y_N$ (where $M \leqslant N \leqslant 2M$).

Let $p$ be bigger than $\max_{\ell=1,\ldots,N} \#\mathcal{O}(y_\ell)$ and such that $y_\ell$ ($\ell = 1, \ldots, N$) is eventually $p$-periodic. Observe that $f^p$ maps the points $y_1, \ldots, y_N$ to fixed points of $f^p$. By possibly going over to multiples of $p$, we may assume without loss of generality that there is $\delta > 0$ such that $f^p$ is $(2\delta)$-locally 3-expanding in a neighborhood of $\bigcup_{\ell=1}^N \mathcal{O}(y_\ell)$. We may further assume $\delta$ to be small enough such that

$$d(f^m(x), f^m(y_\ell)) < 1/2 \cdot \min\left\{\min_{j=1}^M d(\tilde{a}_j, \tilde{b}_j), \varepsilon\right\} = :\varepsilon_0 \qquad (m = 0, \ldots, 2p)$$

whenever $d(x, y_\ell) < \delta$ with $\ell \in \{1, \ldots, N\}$. Choose $n_0$ such that for all $k \geqslant n_0$ we have $d_\infty(f_k^m, f^m) < \delta$ for all $m = 1, \ldots, p$.

Now, $f^p(B_\delta(y_\ell)) \supseteq B_{3\delta}(f^p(y_\ell))$ so that $f_k^p(B_\delta(y_\ell)) \supseteq B_{2\delta}(f^p(y_\ell))$ for $\ell = 1, \ldots, N$ and $k \geqslant n_0$. Similarly, we have $f_k^p(B_\delta(f^p(y_\ell))) \supseteq B_{2\delta}(f^p(y_\ell))$. Altogether, this shows that for all $k \geqslant n_0$ there is $x_\ell^k \in B_\delta(y_\ell)$ with $f_k^p(x_\ell^k) \in B_\delta(f^p(y_\ell))$ and $f_k^p(f_k^p(x_\ell^k)) = f_k^p(x_\ell^k)$.

For $j = 1, \ldots, M$, we define $(a_j^k, b_j^k)$ as follows: if $\tilde{a}_j = y_\ell$ (for some $\ell$) is a surviving endpoint, we set $a_j^k$ to be the right-most point in $\mathcal{O}(x_\ell^k) \cap B_{\varepsilon_0}(\tilde{a}_j)$ and $b_j^k$ the left-most point in $\left(\mathcal{O}(x_\ell^k) \cup \{\tilde{b}_j\}\right) \cap B_{\varepsilon_0}(\tilde{b}_j)$. If $\tilde{a}_j$ is not a surviving endpoint, then $\tilde{b}_j$ is necessarily surviving and we proceed similarly. Note that $(a_j^k, b_j^k) \in \mathcal{B}_{f_k}$ and $d(a_j^k, \tilde{a}_j), d(b_j^k, \tilde{b}_j) < \varepsilon_0 \leqslant \varepsilon/2$ $(j = 1 \ldots, N)$ and hence $\mathcal{B}_f \subseteq \bigcup_{j=1}^{\ell} B_{3\varepsilon}((a_j^k, b_j^k))$. Since $\varepsilon > 0$ can be chosen arbitrarily small, this proves the desired statement. □

### *4.2. Critical steps*

Throughout this section, we consider continuous self-maps $f$ on $\mathbb{I}$ which are piecewise uniformly expanding (for the relation to piecewise monotone maps, see remark 4.12). Recall that $f$ is referred to as *piecewise uniformly expanding*, if there are finitely many intervals $I_1, \ldots, I_n$ with $\mathbb{I} \subseteq \bigcup_{\ell=1}^{n} I_\ell$ such that $f$ is uniformly expanding on each such interval, that is, there is $\lambda > 1$ such that $d(f(x), f(y)) > \lambda \cdot d(x, y)$ whenever there is $\ell$ with $x, y \in I_\ell$. Given these intervals are maximal, the corresponding boundary points which do not lie in $\partial\mathbb{I}$ coincide with the critical points of $f$.

Recall that a double point $(a, b) \in \mathcal{B}_f$ is referred to as periodic, if both $a$ and $b$ are periodic. Our goal is to take a close look at periodic corner points (where $\mathcal{O}(a) = \mathcal{O}(b)$) with $Cri(f) \cap \mathcal{O}(a) \neq \emptyset$. We call this kind of periodic corner points (and hence steps, according to the previous section) *critical*. Given a critical step $(a, b)$ of period $p$, we say that $f$ is *positive at $a$* whenever the image under $f^p$ of arbitrary small closed segments containing $a$ is given by $[f^p(a), c]$ where $\mathbb{I} \ni c \neq f^p(a) = a$. Clearly, if $f$ is not positive at $a$, then the image under $f^p$ of an arbitrary small enough closed interval is given by $[c, f^p(a)]$ for some $c \neq f^p(a) = a$ in $\mathbb{I}$. In this situation, we say that $f$ is *negative at $a$*. For a periodic step $(a, b) \in \mathcal{B}_f$ we say that $f$ is *positive from $a$ to $b$* if $f$ preserves orientation from $a$ to $b$ or if for some small enough segment $J$ containing $a$ in its interior, we have $f^{n_{a,b}}(J) = [b, c]$ where $\mathbb{I} \ni c \neq f^{n_{a,b}}(a) = b$. In the complementary situation, we have that $f$ either reverses orientation from $a$ to $b$ or we have that for an arbitrary small enough segment $J$ containing $a$ in its interior it holds $f^{n_{a,b}}(J) = [c, b]$ for some $c \neq f^{n_{a,b}}(a) = b$ in $\mathbb{I}$. In either case we say that $f$ *is negative from $a$ to $b$*. The following statement shows that in several situations $\mathcal{B}_f$ detects the periodicity of critical points explicitly.

**Lemma 4.6.** *Let $(a, b) \in \mathcal{B}_f$ be a critical step of a transitive piecewise uniformly expanding map $f : \mathbb{I} \to \mathbb{I}$. If $f$ is negative at $a$ and positive at $b$, $(a, b)$ is accumulated from below.*

**Proof.** We first show that for every $\varepsilon > 0$ we have that there is $n \in \mathbb{N}$ and two distinct points $x, y \in [a, a + \varepsilon] = I$ with $f^n(x), f^n(y) \in \mathcal{O}(a)$ (note that possibly $f^n(x) = f^n(y) = a$). To that end, we may assume without loss of generality that $\varepsilon > 0$ is small enough to guarantee that $(a + \varepsilon, b')$ is a non-empty subinterval of $(a, b)$, where $b'$ is the left-most point of $(a, b) \cap (f^{n_{a,b}}(I) \cup \{b\})$. As $f$ is transitive, there clearly exists a transitive point $z \in I$. In particular, there must be $n \geqslant 1$ such that $f^n(z) \in (a + \varepsilon, b')$. Note that this necessarily gives $\{a\} \subseteq f^n(I) \cap \mathcal{O}(a)$ or $\{b\} \subseteq f^n(I) \cap \mathcal{O}(a)$. In the first case, if $n$ is not a multiple of the minimal period $p$ of $a$, we are done since $f^n(a)$ obviously lies in $\mathcal{O}(a)$ which would hence give two points in $\mathcal{O}(a)$. If, however, $n$ is a multiple of $p$, we must have another point besides $a$ whose $n$th image coincides with $a$ as $f$ is assumed to be negative at $a$. The second case can be dealt with similarly.

Now, assume $n \in \mathbb{N}$ to be minimal with the discussed property and observe that the above argument also gives

$$f^j(I) \cap (a, b) = \emptyset \quad \text{for } j = 1, \ldots, n_{a,b} - 1, n_{a,b} + 1, \ldots, n - 1. \tag{3}$$

By definition of $n$, we hence have $x_0 \in I \backslash \{a\}$ with $f^n(x_0) \in \mathcal{O}(a)$ and such that $f^j(x_0) \in \mathbb{I} \backslash (a, b)$ for every $j = 1, \ldots, n_{a,b} - 1, n_{a,b} + 1, \ldots, n - 1$, due to (3). Clearly, given $\delta > 0$ we can further guarantee that $y_0 = f^{n_{a,b}}(x_0)$ is $\delta$-close to $b$ by choosing the above $\varepsilon$ small enough. Then $(x_0, y_0)$ is a double point $2\delta$-close to $(a, b)$ and below $(a, b)$. Since $\delta > 0$ was arbitrary, this proves the statement. $\square$

**Corollary 4.7.** *Let $f : \mathbb{I} \to \mathbb{I}$ be continuous, transitive and piecewise uniformly expanding. Then $\mathcal{B}_f$ has a step $(a, b)$ accumulated from below if and only if $(a, b)$ is a critical step and $f$ is negative at $a$ and positive at $b$.*

**Proof.** The 'if'-part is given by the previous statement. For the other direction consider a periodic corner point $(a, b) \in \mathcal{B}_f$ accumulated from below. If it is hyperbolic, then it cannot be accumulated from below due to lemma 4.1. Hence, it must be a critical periodic corner point. For a contradiction, suppose $f$ is negative at $a$ and negative at $b$ (the other cases can be dealt with similarly) and assume there is $(a', b') \in \mathcal{B}_f$ with $a < a' < a + \delta$ and $b - \delta < b' < b$ for arbitrarily small $\delta > 0$. We denote by $p$ the minimal period of $a$ and $b$. For small enough $\delta > 0$, the negativity at $b$ implies that $f^p(b') \in (a', b')$ and that there is $\ell \in \mathbb{N}$ such that $f^{n_{a,b}+\ell\cdot p}(a') \in (a', b')$ since $f$ is piecewise uniformly expanding. For such $\delta$ we have $(a, a + \delta) \times (b - \delta, b) \subseteq \mathcal{B}_f^c$ which finishes the proof. $\square$

If $\mathbb{I} = \mathbb{T}$, we clearly have that if $b$ is the second coordinate of a step, then it also is the first coordinate of the neighboring step of the associated stair. In this way, we obtain the following statement where the term *negative slope region* of a piecewise uniformly expanding map refers to a maximal interval in the complement of the critical points on which the map reverses orientation. The straightforward proof is left to the reader.

**Corollary 4.8.** *Suppose $f : \mathbb{T} \to \mathbb{T}$ is a transitive piecewise uniformly expanding map. Then there is a step $(a, b)$ in $\mathcal{B}_f$ which is accumulated from below if and only if $f$ has a critical periodic point which meets a negative slope region or it has a critical periodic point with an orbit supporting both a local maximum and a local minimum of $f$.*

It remains to study the case when $(a, b) \in \mathcal{B}_f$ is a critical periodic corner point not fulfilling the conditions of lemma 4.6. In this case, we obtain the following

**Lemma 4.9.** *Let $f : \mathbb{I} \to \mathbb{I}$ be continuous, transitive and piecewise uniformly expanding. Suppose $(a, b) \in \mathcal{B}_f$ is a critical periodic corner point such that $f$ is positive at $a$. Then there is a neighborhood $U \subseteq \Delta$ of $(a, b)$ and a sequence of maps $(f_n)_{n\in\mathbb{N}}$ converging uniformly to $f$ so that $\mathcal{B}_{f_n} \cap U = \emptyset$ for every $n \in \mathbb{N}$. The same holds true if $f$ is negative at $b$.*

**Proof.** Let $f$ be positive at $a$ (the proof of the other case works similarly) and let $p$ denote the minimal period of $a$. As $f$ is piecewise uniformly expanding and positive at $a$, there are $\varepsilon_1, \varepsilon_2, \delta > 0$ such that for $I = (a - \varepsilon_1, a + \varepsilon_2)$, $I^- = (a - \varepsilon_1, a)$ and $I^+ = (a, a + \varepsilon_2)$

(a) $f^p(I^-) = f^p(I^+) = (a, a + \delta)$,
(b) $f^p$ is uniformly expanding on $I^+$,
(c) $f^j(I) \cap [a - \varepsilon_1 - \delta, a + \varepsilon_2 + \delta] = \emptyset$ for $j = 1, \ldots, p - 1$.

Observe the following: given $t \in (0, \varepsilon_2)$, for every $x \in I$ there exit $m_x \in \mathbb{N}$ such that $(f^p + t)^{m_x}(x) \in [a + \varepsilon_2, a + \varepsilon_2 + \delta) = J$.

To see this, note that for $x \in I$ with $y = (f^p + t)(x) \notin J$ we have $y \in I^+$, due to (a). Since $f^p + t$ is uniformly expanding on $I^+$ (by (b)), the existence of the above $m_x$ follows.

Now, for big enough $n \in \mathbb{N}$, there is an orientation preserving homeomorphism $g_n$ with $g_n = \mathrm{Id}$ on $\mathbb{I}\backslash[a - \varepsilon_1 - \delta, a + \varepsilon_2 + \delta]$, $g_n(x) = x + 1/n$ on $(a - \varepsilon_1, a + \delta)$ and $d_\infty(g_n, \mathrm{Id}) = 1/n$. Set $f_n = g_n \circ f$. On the interval $I$, we have $(g_n \circ f)^p = (f^p + 1/n)$, due to (a) and (c). If $1/n < \varepsilon_2$, the above observation implies that for every $x \in I$ we have $m_x \in \mathbb{N}$ such that $(g_n \circ f)^{p\cdot m_x}(x) = (f^p + 1/n)^{m_x}(x) \in J$.

Consider now a neighborhood $U$ of $(a,b) \in \Delta$ such that for $(a',b') \in U$ we have $a' \in I$, $(a',b') \supset J$, and $f^{n_{b,a}}(b') \in I$. Then, given $(a',b') \in U$ we have for large enough $n$ that $(g_n \circ f)^{p \cdot m_{a'}}(a') \in (a',b')$ and $(g_n \circ f)^{p \cdot m_z}(z) \in (a',b')$ where $z = f^{n_{b,a}}(b')$. Hence, $U \subseteq \Delta \backslash \mathcal{B}_{g_n \circ f}$ for big enough $n$ which proves the statement. □

For the next statement, we consider the space of continuous self-maps on $\mathbb{I}$ equipped with the supremum metric $d_\infty$ and the space of non-empty closed subsets of $\Delta$ endowed with the Hausdorff metric.

**Corollary 4.10.** *Let $f : \mathbb{I} \to \mathbb{I}$ be a continuous transitive piecewise uniformly expanding map and $(a,b) \in \mathcal{B}_f$ a critical periodic corner point such that f is positive at a or negative at b. Then the map $g \mapsto \mathcal{B}_g$ is not continuous at f.*

Summing-up, we obtain the following statement concerning the sensitivity of the bifurcation set to different dynamical behavior of the critical points.

**Theorem 4.11.** *Assume that $f : \mathbb{I} \to \mathbb{I}$ is a continuous transitive piecewise uniformly expanding map. The following holds.*

(*a*) *If* $\mathrm{Per}(f) \cap \mathrm{Cri}(f) = \emptyset$, *then every step is isolated from below. Further, in case* $\mathrm{Cri}(f)$ *is empty or only consists of transitive points, we get that f is a continuity point of the map* $g \mapsto \mathcal{B}_g$.

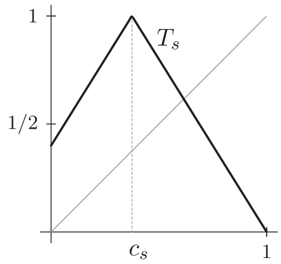
(*b*) *If* $\mathrm{Per}(f) \cap \mathrm{Cri}(f) \neq \emptyset$, *then there is at least one step accumulated from below or f is a discontinuity point of* $g \mapsto \mathcal{B}_g$.

**Remark 4.12.** According to a classical result of Parry [31, corollary 3], a transitive piecewise monotone map $f : [0,1] \to [0,1]$ is conjugate to a map of constant slope $\pm\beta$ where $\log \beta$ is the topological entropy of $f$. Further, it is well known that transitive continuous interval maps have positive entropy, see for instance [8, corollary 3.6]. Therefore, we can use theorem 4.11 to infer theorem B for interval maps because topological properties of the bifurcation set are preserved under conjugation, see section 2.1. Concerning maps on the circle, [3, theorem C] provides the analogue statement of Parry's result. Moreover, for the fact that transitive non-minimal circle maps have positive entropy, see for example [2, p 267].

## 5. Proof of theorem C

In order to emphasize the applicability of our results, we now make use of the statements and techniques of the previous sections to describe the dependence of the bifurcation set on the parameter of a particular family of interval maps. The specific family we are interested in is given by the collection of *restricted tent maps* $(T_s)_{s \in (1,2]}$ which are defined via

$$T_s(x) := \begin{cases} 1 + s(x - c_s) & \text{if } x \in [0, c_s] \\ 1 - s(x - c_s) & \text{if } x \in (c_s, 1] \end{cases} \quad \text{with } c_s := 1 - \frac{1}{s}.$$




The above figure depicts $T_s$ for $s = 1.6$. It is not difficult to show that each $T_s$ is conjugate to the tent map $t_s : [0,1] \to [0,1]$ given by $x \mapsto s(1/2 - |x - 1/2|)$ restricted to the interval

$[t_s^2(1/2), t_s(1/2)]$ for $s \in (1, 2]$. Moreover, it is well known that $T_s$ is transitive if and only if $s \in [\sqrt{2}, 2]$, see e.g. [34, lemma 8.1]. Using the relation between $T_s$ and $t_s$, the following holds.

**Theorem 5.1 ([9, theorem 7] and [14, lemma 5.5]).** *For almost every $s \in [\sqrt{2}, 2]$ we have that the critical point $c_s$ is transitive. Further, $c_s$ is a periodic point for a dense set of parameters.*

By means of this result, we will obtain

**Theorem 5.2.** *Consider the family of restricted tent maps $(T_s)_{s\in[\sqrt{2},2]}$. Then*

(*a*) *If $c_{s_0}$ is transitive, the steps of $\mathcal{B}_{T_{s_0}}$ are isolated from below and the map $s \mapsto \mathcal{B}_{T_s}$ is continuous at $s_0$.*

(*b*) *If $c_{s_0}$ is periodic, the map $s \mapsto \mathcal{B}_{T_s}$ is not continuous at $s_0$.*

(*c*) *There exists a dense set of parameters $\mathcal{J} \subseteq [\sqrt{2}, 2]$ so that $c_s$ is periodic and there are steps of $\mathcal{B}_{T_s}$ which are accumulated from below whenever $s \in \mathcal{J}$.*

We devote the rest of the section to the proof of this statement. Clearly, point (a) follows from theorem 4.11. In view of lemma 4.6, point (c) can be deduced from the next proposition.

**Proposition 5.3.** *There is a dense set of parameters $\mathcal{J} \subset [\sqrt{2}, 2]$ such that for every $s \in \mathcal{J}$ we have a step $(a, b) \in \mathcal{B}_{T_s}$ where $T_s$ is negative at $a$ and positive at $b$.*

**Proof.** Observe that for all $s \in (1, 2]$ the point $x_s = 1 - 1/s - 1/s^2 + 1/s^3$ verifies $x_s \in (0, c_s)$, $T_s(x_s) \in (c_s, 1)$ and $T_s^2(x_s) = c_s$. By choosing $y_s$ sufficiently close and to the right of $x_s$, we can guarantee that $0 < y_s < T_s^2(y_s) < c_s < T_s(y_s) < T_s^3(y_s) < 1$. In particular, $T_s$ is order preserving in $y_s$ as well as in $T_s^2(y_s)$ and order reversing in $T_s(y_s)$ as well as in $T_s^3(y_s)$.

Given $s \in [\sqrt{2}, 2]$, by theorem 5.1 there is an arbitrarily close $s'$ such that $c_{s'}$ is transitive. Observe that there is $n \in \mathbb{N}$ with $0 < T_{s'}^n(c_{s'}) < T_{s'}^{n+2}(c_{s'}) < c_{s'} < T_{s'}^{n+1}(c_{s'}) < T_{s'}^{n+3}(c_{s'}) < 1$ (pick $n$ such that $T_{s'}^n(c_{s'})$ is sufficiently close to $y_{s'}$). Now, theorem 5.1 allows to pick $s''$ such that $c_{s''}$ is periodic and such that $s''$ is sufficiently close to $s'$ to guarantee $0 < T_{s''}^n(c_{s''}) < T_{s''}^{n+2}(c_{s''}) < c_{s''} < T_{s''}^{n+1}(c_{s''}) < T_{s''}^{n+3}(c_{s''}) < 1$. Hence, at $s''$ we have $a' < b' < c' < d'$ where $a' = T_{s''}^n(c_{s''})$, $b' = T_{s''}^{n+2}(c_{s''})$, $c' = T_{s''}^{n+1}(c_{s''})$ and $d' = T_{s''}^{n+3}(c_{s''})$. Note that either $T_{s''}$ is negative at $a'$, positive at $b'$, negative at $c'$ and positive at $d'$ or the other way around, that is, $T_{s''}$ is positive at $a'$, negative at $b'$, positive at $c'$ and negative at $d'$.

In the first case, choose $a \in \mathcal{O}(c_{s''}) \cap [a', b']$ to be such that $T_{s''}$ is negative at $a$ and $T_{s''}$ is positive at each element of $(a, b'] \cap \mathcal{O}(c_{s''})$. Choose $b$ to be the smallest element of $(a, b'] \cap \mathcal{O}(c_{s''})$. Then $(a, b)$ is a periodic corner point with $T_{s''}$ negative at $a$ and positive at $b$. In the second case (when $T_{s''}$ is positive at $a$ etc), we obtain a similar statement by dealing with $b'$ instead of $a'$ and $c'$ instead of $b'$.

As $s \in [\sqrt{2}, 2]$ is arbitrary and $s''$ can be chosen arbitrarily close to $s$, the statement follows. $\square$

With regards to point (b) of theorem 5.2, observe that if $c_s$ is periodic, then $0 = T_s^2(c_s)$ is periodic, too. Therefore, as $T_s$ is clearly positive at 0, lemma 4.9 yields that $T_s$ is a discontinuity point of the mapping $f \mapsto \mathcal{B}_f$.[8] In proposition 5.5 (see below), we will show that this discontinuity is already visible within the family $(T_s)_{s\in[\sqrt{2},2]}$. To see this, we first make some simple technical observations.

[8] Note that formally speaking, as our present definition of $\mathcal{B}_f$ excludes points with coordinate entries equal to zero, we could not apply lemma 4.9. However, this issue is of a rather formal nature (see also the remark in the introduction) and will further not play a role in the discussion of the discontinuity of $s \mapsto \mathcal{B}_{T_s}$ as this discussion has to be carried out explicitly anyway.

Note that for all $x \in [0,1]$ and all $\ell \in \mathbb{N}$ with $T_s^j(x) \neq c_s$ ($j = 0, \ldots, \ell$), we have that $T_s^{\ell+1}(x)$ is differentiable with respect to $s$ (as well as $x$) and

$$\frac{\mathrm{d}}{\mathrm{d}s} T_s^{\ell+1}(x) = \left(\frac{\partial}{\partial s} T_s\right) \left(T_s^\ell(x)\right) + T_s' \left(T_s^\ell(x)\right) \cdot \frac{\mathrm{d}}{\mathrm{d}s} T_s^\ell(x).$$

As

$$T_s'(x) = \begin{cases} s & \text{if } x \in [0, c_s) \\ -s & \text{if } x \in (c_s, 1] \end{cases} \quad \text{and} \quad \frac{\partial}{\partial s} T_s(x) = \begin{cases} -1 + x & \text{if } x \in [0, c_s) \\ 1 - x & \text{if } x \in (c_s, 1] \end{cases}, \tag{4}$$

we hence have

$$\frac{\mathrm{d}}{\mathrm{d}s} T_s^{\ell+1}(x) = \begin{cases} -1 + T_s^\ell(x) + s \cdot \dfrac{\mathrm{d}}{\mathrm{d}s} T_s^\ell(x) & \text{if } T_s^\ell(x) \in [0, c_s) \\ 1 - T_s^\ell(x) - s \cdot \dfrac{\mathrm{d}}{\mathrm{d}s} T_s^\ell(x) & \text{if } T_s^\ell(x) \in (c_s, 1] \end{cases}. \tag{5}$$

**Proposition 5.4.** *Consider the family of restricted tent maps $(T_s)_{s \in [\sqrt{2}, 2]}$. If $c_{s_0}$ is periodic, then $\frac{\mathrm{d}}{\mathrm{d}s}(T_s^{n_{0,c_{s_0}}}(0) - c_s) \neq 0$ at $s = s_0$.*

*Moreover, if $\frac{\mathrm{d}}{\mathrm{d}s}(T_s^{n_{0,c_{s_0}}}(0) - c_s) > 0$, then $\left(T_s^{n_{0,c_{s_0}}}\right)'(0) = -s^{n_{0,c_{s_0}}}$. Otherwise $\left(T_s^{n_{0,c_{s_0}}}\right)'(0) = s^{n_{0,c_{s_0}}}$.*

**Proof.** First, note that $T_{s_0}^j(0) \neq c_{s_0}$ for all $j = 0, \ldots, n_{0,c_{s_0}} - 1$ so that the above expressions are indeed differentiable. For the first part, we have to consider three cases.

Case 1: $s_0 = (1 + \sqrt{5})/2$. This is the only case in which $n_{0,c_{s_0}} = 1$. By (4), we have $\frac{\mathrm{d}}{\mathrm{d}s}(T_s(0) - c_s) = -1 - 1/s^2 < 0$. In the remaining cases, we will show that, in fact,

$$\left|\frac{\mathrm{d}}{\mathrm{d}s} T_s^{n_{0,c_{s_0}}}(0)\right| \geqslant 1/(s-1) > \left|\frac{\mathrm{d}}{\mathrm{d}s} c_s\right| = 1/s^2 \tag{6}$$

at $s = s_0$. To that end, note that (5) yields that if

$$\left|\frac{\mathrm{d}}{\mathrm{d}s} T_s^j(0)\right| \geqslant 1/(s-1) \tag{7}$$

for some $j = j_0 < n_{0,c_{s_0}}$, then (7) also holds for $j = j_0 + 1$. Hence, it suffices to show that there is some $j \leqslant n_{0,c_{s_0}}$ for which (7) holds in order to prove (6) and hence the first part of the statement.

Case 2: $s_0 \in [3/2, 2] \setminus \{(1 + \sqrt{5})/2\}$. By an immediate computation, we have $\left|\frac{\mathrm{d}}{\mathrm{d}s} T_s^2(0)\right| = 2s - 1$ at all $s \in [\sqrt{2}, 2]$ and hence $\left|\frac{\mathrm{d}}{\mathrm{d}s} T_s^2(0)\right| \geqslant 1/(s-1)$ for all $s \geqslant 3/2$.

Case 3: $s_0 < 3/2$. Observe that in this case we have $T_{s_0}(0), T_{s_0}^2(0), T_{s_0}^3(0) \in (c_{s_0}, 1]$. It hence suffices to show (7) for $j = 4$. Now, if $T_s(0), T_s^2(0), T_s^3(0) \in (c_s, 1]$, we have $T_s^4(0) = s^4 - s^3 - s^2 + s$. Therefore, $\frac{\mathrm{d}}{\mathrm{d}s} T_s^4(0) = 4s^3 - 3s^2 - 2s + 1$. By elementary means, we see that this indeed gives $\frac{\mathrm{d}}{\mathrm{d}s} T_s^4(0) \geqslant 1/(s-1)$ for all $s < 3/2$ which finishes the proof of the first part.

For the second part, notice that if $\left|\frac{\mathrm{d}}{\mathrm{d}s} T_s^j(0)\right| \geqslant 1/(s-1)$ for some $j < n_{0,c_s}$ and $\frac{\mathrm{d}}{\mathrm{d}s} T_s^j(0)$ is positive (negative), then (5) gives that $\frac{\mathrm{d}}{\mathrm{d}s} T_s^{j+1}(0)$ is positive (negative) if and only if

$T_s^j(0) \in [0, c_s)$. With this in mind, an inspection of the above cases shows that whenever we have $\frac{\mathrm{d}}{\mathrm{d}s}(T_s^{n_{0,c_{s_0}}}(0) - c_s) < 0$ it holds that $\#\{j \in \{0, \ldots, n_{0,c_{s_0}} - 1\} : T_{s_0}^j(0) \in (c_{s_0}, 1]\}$ is even. Hence, in this case we obtain by the chain rule that $\left(T_s^{n_{0,c_{s_0}}}\right)'(0) = T_s'\left(T_s^{n_{0,c_{s_0}}-1}(0)\right) \cdot \ldots \cdot T_s'(0) = s^{n_{0,c_{s_0}}}$. The other case is similar. $\square$

**Proposition 5.5.** *Consider the family of restricted tent maps $(T_s)_{s \in [\sqrt{2},2]}$. If $c_{s_0}$ is periodic, then the map $s \mapsto \mathcal{B}_{T_s}$ is not continuous at $s_0$.*

**Proof.** Let $p$ be the minimal period of 0 under $T_{s_0}$ and let $b$ be the element in $\mathcal{O}(0)\backslash\{0\}$ which is the closest to 0. Note that the horizontal segment $H = \{(a, b) : 0 < a < b\}$ is entirely contained in $\mathcal{B}_{T_{s_0}}$. Our goal is to show that there is some $\varepsilon_0 > 0$ such that $(0, \varepsilon_0) \times B_{\varepsilon_0}(b) \subseteq \mathcal{B}_{T_s}^c$ for $s$ sufficiently close to $s_0$. This clearly proves the statement.

By proposition 5.4, we either have $\frac{\mathrm{d}}{\mathrm{d}s}(T_s^{n_{0,c_{s_0}}}(0) - c_s) > 0$ and $\left(T_s^{n_{0,c_{s_0}}}\right)'(0) = -s^{n_{0,c_{s_0}}}$ or $\frac{\mathrm{d}}{\mathrm{d}s}(T_s^{n_{0,c_{s_0}}}(0) - c_s) < 0$ and $\left(T_s^{n_{0,c_{s_0}}}\right)'(0) = s^{n_{0,c_{s_0}}}$. Set $\mathcal{I}' = (s_0 - \delta, s_0)$ for some $\delta > 0$.[9] Observe that $T_s^j(x) \neq c_s$ $(x \in [0, \varepsilon], \quad s \in \mathcal{I}', \quad j = 0, \ldots, p)$ and hence, in fact, $T_s^p(x) - T_s^p(y) = s^p(x - y)$ for all $x, y \in [0, \varepsilon]$ and each $s \in \mathcal{I}'$ whenever $\varepsilon > 0$ and $\delta$ are sufficiently small.

W.l.o.g. we may assume $\varepsilon < b/(4s^p)$ as well as $T_s^p(0), T_s^{n_{b,0}}(b) < \varepsilon/2$ $(s \in \mathcal{I}')$, where $n_{b,0} < p$ is such that $T_{s_0}^{n_{b,0}}(b) = 0$. Clearly, $T_s^p(x) = T_s^p(0) + s^p x < \varepsilon + s^p\varepsilon/2 < b/2$ whenever $x \in [0, \varepsilon]$.

Altogether, the above shows that for each $x \in (0, \varepsilon]$ and every $s \in \mathcal{I}'$ we have some $\ell \in \mathbb{N}$ with $T_s^{\ell p}(x) \in (\varepsilon, b/2)$. Accordingly, if $|x| < \varepsilon$ and $|b - y| < \varepsilon/(2s_0^{n_{b,0}})$, we have some $\ell_x, \ell_y \in \mathbb{N}$ such that $T_s^{\ell_x p}(x), T_s^{\ell_y p}(y) \in (\varepsilon, b/2)$ and hence $(x, y) \in \mathcal{B}_{T_s}^c$ for every $s \in \mathcal{I}'$. Therefore, $\mathcal{B}_{T_s} \cap \left((0, \varepsilon) \times B_{\varepsilon/(2s_0^{n_{b,0}})}(b)\right) = \emptyset$ for each $s \in \mathcal{I}'$. $\square$

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## References

[1] Agarwal N 2020 The $k$-transformation on an interval with a hole *Qual. Theory Dyn. Syst.* **19** 30

[2] Alsedà L, Llibre J and Misiurewicz M 2000 *Combinatorial Dynamics and Entropy in Dimension One* (Singapore: World Scientific)

[3] Alsedà L and Misiurewicz M 2015 Semiconjugacy to a map of a constant slope *Discrete Contin. Dyn. Syst.* B **20** 3403–13

[9] Note that $s_0$ is necessarily different from $\sqrt{2}$, since $T_{\sqrt{2}}(0)$ coincides with the unique fixed point of $T_{\sqrt{2}}$ which gives that $c_{\sqrt{2}}$ is not periodic. Hence, $\mathcal{I}'$ is always an interval which has a non-trivial intersection with $[\sqrt{2}, 2]$.

[4] Altmann E G, Portela J S E and Tél T 2013 Leaking chaotic systems *Rev. Mod. Phys.* **85** 869–918
[5] Auslander J and Katznelson Y 1979 Continuous maps of the circle without periodic points *Isr. J. Math.* **32** 375–81
[6] Baker S and Kong D 2020 Two bifurcation sets arising from the beta transformation with a hole at 0 *Indagat. Math.* **31** 436–49
[7] Block L and Coppel W 1992 *Dynamics in One Dimension* (*Lecture Notes in Mathematics*) (Berlin: Springer)
[8] Block L and Coven E M 1987 Topological conjugacy and transitivity for a class of piecewise monotone maps of the interval *Trans. Am. Math. Soc.* **300** 297
[9] Brucks K and Misiurewicz M 1996 The trajectory of the turning point is dense for almost all tent maps *Ergod. Theory Dyn. Syst.* **16** 1173–83
[10] Bruin H, Demers M F and Todd M 2018 Hitting and escaping statistics: mixing, targets and holes *Adv. Math.* **328** 1263–98
[11] Bunimovich L A and Yurchenko A 2011 Where to place a hole to achieve a maximal escape rate *Isr. J. Math.* **182** 229–52
[12] Carminati C and Tiozzo G 2011 The bifurcation locus for numbers of bounded type (arXiv:1109.0516)
[13] Clark L 2016 The $\beta$-transformation with a hole *Discrete Contin. Dyn. Syst.* A **36** 1249–69
[14] Coven E M, Kan I and Yorke J A 1988 Pseudo-orbit shadowing in the family of tent maps *Trans. Am. Math. Soc.* **308** 227
[15] Coven E M and Mulvey I 1986 Transitivity and the centre for maps of the circle *Ergod. Theory Dyn. Syst.* **6** 1–8
[16] de Melo W and van Strien S 1993 *One-Dimensional Dynamics* (*A Series of Modern Surveys in Mathematics* vol 25) (Berlin: Springer)
[17] Demers M and Fernandez B 2016 Escape rates and singular limiting distributions for intermittent maps with holes *Trans. Am. Math. Soc.* **368** 4907–32
[18] Demers M F and Young L-S 2005 Escape rates and conditionally invariant measures *Nonlinearity* **19** 377–97
[19] Dettmann C 2011 *Recent Advances in Open Billiards with Some Open Problems* (*World Scientific Series on Nonlinear Science* vol 16) (Singapore: World Scientific) pp 195–218
[20] Glendinning P and Sidorov N 2015 The doubling map with asymmetrical holes *Ergod. Theory Dyn. Syst.* **35** 1208–28
[21] Glendinning P and Sparrow C 1993 Prime and renormalisable kneading invariants and the dynamics of expanding Lorenz maps *Physica* D **62** 22–50
[22] Gröger M 2015 Examples of dynamical systems in the interface between order and chaos *PhD Thesis* Universität Bremen, Bremen
[23] Hare K G and Sidorov N 2014 On cycles for the doubling map which are disjoint from an interval *Monatsh. Math.* **175** 347–65
[24] Hubbard J H and Sparrow C T 1990 The classification of topologically expansive Lorenz maps *Commun. Pure Appl. Math.* **43** 431–43
[25] Jensen J 2011 On lower bounded orbits of the times-$Q$ map *Unif. Distrib. Theory* **6** 157–75
[26] Kalle C, Kong D, Langeveld N and Li W 2020 The $\beta$-transformation with a hole at 0 *Ergod. Theory Dyn. Syst.* **9** 2482–514
[27] Keller G and Liverani C 2009 Rare events, escape rates and quasistationarity: some exact formulae *J. Stat. Phys.* **135** 519–34
[28] Martens M 1994 Distortion results and invariant Cantor sets of unimodal maps *Ergod. Theory. Dyn. Syst.* **14** 331–49
[29] Misiurewicz M and Szlenk W 1980 Entropy of piecewise monotone mappings *Stud. Math.* **67** 45–63
[30] Nilsson J 2009 On numbers badly approximable by dyadic rationals *Isr. J. Math.* **171** 93–110
[31] Parry W 1966 Symbolic dynamics and transformations of the unit interval *Trans. Am. Math. Soc.* **122** 368
[32] Pollicott M and Urbański M 2017 *Open Conformal Systems and Perturbations of Transfer Operators* (Berlin: Springer)
[33] Sharkovsky A 1964 Fixed points and the center of a continuous mapping of the line into itself (Ukrainian) *Dopovidi Akademiï Nauk Ukraïns'koï RSR* vol 7 pp 865–8
[34] Shultz F 2007 Dimension groups for interval maps II: the transitive case *Ergod. Theory Dyn. Syst.* **27** 1287–321

[35] Sidorov N 2014 Supercritical holes for the doubling map *Acta Math. Hung.* **143** 298–312
[36] Urbański M 1986 On Hausdorff dimension of invariant sets for expanding maps of a circle *Ergod. Theory Dyn. Syst* **6** 295–309
[37] Urbański M 1987 Invariant subsets of expanding mappings of the circle *Ergod. Theory Dyn. Syst.* **7** 627–45